

Quantum Geons and Noncommutative Spacetimes

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Abstract

Dedication: *This article is dedicated to Josh Goldberg in appreciation of his friendship and his long years of service and contributions to the field of relativity.*

Physical considerations strongly indicate that spacetime at Planck scales is noncommutative. A popular model for such a spacetime is the Moyal plane. The Poincaré group algebra acts on it with a Drinfel'd-twisted coproduct, however the latter is not appropriate for more complicated spacetimes such as those containing Friedman-Sorkin (topological) geons. They have rich diffeomorphisms and mapping class groups, so that the statistics groups for N identical geons is strikingly different from the permutation group S_N . We generalise the Drinfel'd twist to (essentially all) generic groups including finite and discrete ones, and use it to deform the commutative spacetime algebras of geons to noncommutative algebras. The latter support twisted actions of diffeomorphisms of geon spacetimes and their associated twisted statistics. The notion of covariant quantum fields for geons is formulated and their twisted versions are constructed from their untwisted counterparts. *Non-associative* spacetime algebras arise naturally in our analysis. Physical consequences, such as the violation of Pauli's principle, seem to be one of the outcomes of such nonassociativity.

The richness of the statistics groups of identical geons comes from the nontrivial fundamental group of their spatial slices. As discussed long ago, extended objects like rings and D -branes also have similar rich fundamental groups. This work is recalled and its relevance to the present quantum geon context is pointed out.

Keywords: Quantum Field Theory, Noncommutative spacetime, Quantum Geons

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I. INTRODUCTION

Attempts to localise Planck-scale spacetime volumes cause the formation of trapped surfaces and black holes by the principles of quantum theory and relativity. Therefore, as argued by Doplicher, Fredenhagen and Roberts (DFR) [1], general considerations suggest a limitation on the precision of spacetime measurements.

Similar limitations in quantum theory such as the Heisenberg uncertainty relations $\Delta x \Delta p \gtrsim \hbar/2$ are accounted for by imposing commutation relations like $[x, p] = i\hbar$. We may thus speculate that limitations on spacetime measurements too can be incorporated by deforming the commutative algebra of functions on spacetime into a noncommutative algebra. For this algebra, for spatial dimension $d = 3$, rigorous four-volume “uncertainty relations” have in fact been found by Doplicher *et al* [2].

When spacetime is \mathbb{R}^{d+1} , a popular deformation is the Moyal algebra $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ where the antisymmetric matrix $\theta = [\theta_{\mu\nu}]$ with constant real coefficients is the deformation parameter. Thus, the coordinate functions \hat{x}_μ in the algebra satisfy the commutation relations $[\hat{x}_\mu, \hat{x}_\nu] = i\theta_{\mu\nu}$.

Remarkably, despite the appearance of the constant matrix θ , the group algebra $\mathbb{C}\mathcal{P}$ of the Hopf-Poincaré group \mathcal{P} acts on $\mathcal{A}_\theta(\mathbb{R}^{d+1})$ as a Hopf algebra $H_\theta\mathcal{P}$ with coproduct [3–5]

$$\Delta_\theta(g) = F_\theta^{-1}(g \otimes g)F_\theta, \quad F_\theta = e^{-\frac{i}{2}P_\mu \otimes \theta^{\mu\nu} P_\nu} \in \mathbb{C}\mathcal{P} \otimes \mathbb{C}\mathcal{P} \quad , \quad (1)$$

$$P_\mu = \text{translation generators} \quad . \quad (2)$$

The Moyal plane with its Hopf symmetry $H_\theta\mathcal{P}$ is a very particular deformation based on the Drinfel’d twist $F_\theta \in \mathbb{C}\mathcal{P} \otimes \mathbb{C}\mathcal{P}$. It has the important physical property that its deformed quantum field theories (qft’s) can be obtained from the undeformed qft’s based on the commutative algebra $\mathcal{A}_0(\mathbb{R}^{d+1})$, the map from one to the other involving a known “dressing transformation” [6, 7]. Generic deformations do not support symmetries of $\mathcal{A}_0(\mathbb{R}^{d+1})$ in any sense. Even when they do, they need not admit unitary qft’s. A simple example is the Wick-Voros plane [8, 9].

The spatial slice \mathbb{R}^d is not the only admissible spatial slice for asymptotically flat spacetimes. Friedman and Sorkin [10] have studied generic asymptotically flat spatial slices and have come up with their remarkable interpretation in terms of gravitational topological excitations called “topological” or “Friedman-Sorkin” “geons”. The diffeomorphisms (diffeos) of geon spacetimes are much richer than those from the topologically

trivial ones. In particular, they contain discrete subgroups encoding the basic physics of geons. It was a striking discovery of Friedman and Sorkin that the geon spin even in pure gravity can be $1/2$ or its odd multiples [10–13]. The statistics groups of identical geons are also novel. Their precise identification requires further considerations as we shall see.

In this paper, we develop a machinery to construct Drinfel’d twists for generic and in particular discrete diffeos. We then recall the notion of covariant quantum fields well-studied for $\mathcal{A}_0(\mathbb{R}^{d+1})$ and extend them for generic spacetimes [14]. This helps us construct covariant twisted fields for geons using the above twists. The requirement of covariance puts conditions on acceptable twists for quantum fields and eliminates many. Previously it eliminated the Wick-Voros twist [8, 9].

Spacetimes emergent from these twists are noncommutative as is appropriate at geon scales according to DFR [1]. There is a diffeo-invariant way to define the size of a geon [15] and it is expected to be of Planck-scale. Spacetime noncommutativity emergent from our approach is localised at geons and is of this scale just as we wish for.

As we indicate, several novel spacetimes including non-associative spacetimes and new sorts of statistics algebras are indicated by this work. In this paper, these matters are discussed only in a preliminary manner. But already, new phenomena like non-Pauli transitions are suggested as we will see.

It was recognised long ago [15, 16] that the novel statistics groups of geons are a reflection of their being extended excitations like solitons, and that similar statistics groups occur for other extended objects like rings and D -branes. Our geon considerations can be adapted to their cases too. We recall these older results briefly towards the end of the paper. Maybe, they will allow explicit computations of physical effects from Drinfel’d twists, and give us indications on what to expect for geons. Detailed calculations with geons are hard as they involve quantum gravity (see however [10, 11]).

II. WHAT ARE GEONS

This is a short review section on the topology of low-dimensional manifolds leading up to those which support geons (“geon manifolds”). The original literature is best consulted for detailed information [10–13].

Given two closed (compact and boundary-less) manifolds \mathcal{M}_1 and \mathcal{M}_2 of dimension d , their connected sum $\mathcal{M}_1 \# \mathcal{M}_2$ is defined as follows. Remove two balls \mathcal{B}_1 and \mathcal{B}_2 from \mathcal{M}_1 and \mathcal{M}_2 , leaving two manifolds

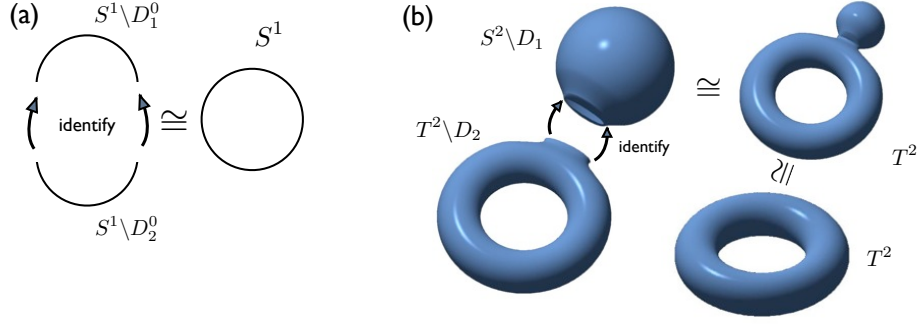


FIG. 1. (a) $S^1 \# S^1 \cong S^1$. (b) $T^2 \# S^2 \cong T^2 \cong S^2 \# T^2$

$\mathcal{M}_i \setminus \mathcal{B}_i$ with spheres S_i^{d-1} ($S_i^{d-1} \sim S^{d-1}$) as boundaries $\partial(\mathcal{M}_i \setminus \mathcal{B}_i)$. Then $\mathcal{M}_1 \# \mathcal{M}_2$ is obtained by identifying these spheres. If \mathcal{M}_i are oriented, this identification must be done with orientation-reversal so that $\mathcal{M}_1 \# \mathcal{M}_2$ is oriented.

Connected summing, $\#$, is associative and commutative:

- a) $\mathcal{M}_1 \# (\mathcal{M}_2 \# \mathcal{M}_3) \cong (\mathcal{M}_1 \# \mathcal{M}_2) \# \mathcal{M}_3$ so that we can write $\mathcal{M}_1 \# \mathcal{M}_2 \# \mathcal{M}_3$;
- b) $\mathcal{M}_1 \# \mathcal{M}_2 \cong \mathcal{M}_2 \# \mathcal{M}_1$.

Here are some simple examples:

- $d = 1$. $S^1 \# S^1 \cong S^1$. (See Fig. 1 (a)).
- $d = 2$. $S^2 \# S^2 \cong S^2$.
- $d = 2$. $T^2 \# S^2 \cong T^2 \cong S^2 \# T^2$. (See Fig. 1 (b)).
- $d = 2$. $T^2 \# T^2 \cong \Sigma_2$. Genus two manifold. (See Fig. 2).

As the examples here suggest, for any dimension d , $\mathcal{M} \# S^d \cong S^d \# \mathcal{M} \cong \mathcal{M}$.

These considerations can be extended to asymptotically flat manifolds. If \mathcal{M}_1 is asymptotically flat and \mathcal{M}_2 is closed and both are oriented (and of the same dimension), then $\mathcal{M}_1 \# \mathcal{M}_2$ is obtained by removing balls \mathcal{B}_i from \mathcal{M}_i and identifying the boundaries $\partial(\mathcal{M}_i \setminus \mathcal{B}_i)$ compatibly with orientation as pointed out above. The connected sum $\mathcal{M}_1 \# \mathcal{M}_2$ is asymptotically flat and oriented.

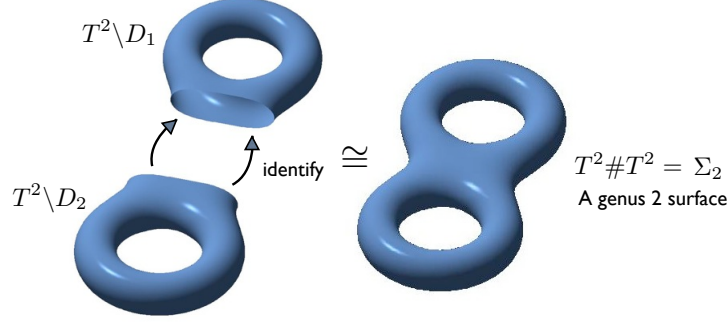


FIG. 2. $T^2 \# T^2 \cong \Sigma_2$. Genus 2 surface.

We will now state certain basic results in low-dimensional topology considering only closed or asymptotically flat, and oriented manifolds \mathcal{M} . In the asymptotically flat case, we will insist that there is only one asymptotic region. That is, the asymptotic region of \mathcal{M} is homeomorphic to the complement of a ball \mathcal{B}^d in \mathbb{R}^d . In other words \mathcal{M} has one asymptotic region if all its topological complexities can be encompassed within a sphere $S^{d-1} \subset \mathcal{M}$.

The case $d = 1$ is trivial, there being only two such manifolds S^1 and \mathbb{R}^1 . (\mathbb{R}^1 has “one” asymptotic region in the above sense even though it is not connected.)

The basic results of interest for $d = 2$ and 3 are as follows.

A. Closed Manifolds

In $d = 2$ and 3, there is a class of special closed manifolds called *prime* manifolds. Any closed manifold $\mathcal{M} \neq S^d$ for $d = 2$ or 3 is a *unique* connected sum of prime manifolds \mathcal{P}_α (with the understanding that spheres are not inserted in the connected sum):

$$\mathcal{M} = \#_\alpha \mathcal{P}_\alpha. \quad (3)$$

(All manifolds have the same dimension. If $\mathcal{M} = S^d$, then (3) is substituted by the triviality $S^d = S^d$, hence, a better way to write (3) is $\mathcal{M} = \#_\alpha \mathcal{P}_\alpha \bmod S^d$.)

The uniqueness of (3) implies that a prime \mathcal{P}_α cannot be decomposed as the connected sum of two or more primes. (It is indecomposable just like a stable elementary particle.)

For $d = 2$, there is just one prime, namely the torus. In that case, $T^2 \# T^2 \# \dots \# T^2$ with k terms is just a

genus k surface (see Fig. 2 for $k = 2$).

For $d = 3$, there are an infinity of prime manifolds. They are not fully known. Representative examples are the following:

a) *Spherical Space Forms*. Notice that $S^3 \cong SU(2)$ by writing

$$SU(2) \ni g = \begin{pmatrix} \xi_1 & -\bar{\xi}_2 \\ \xi_2 & \bar{\xi}_1 \end{pmatrix}, \quad |\xi_1|^2 + |\xi_2|^2 = 1. \quad (4)$$

Then $SO(4) = \frac{SU(2) \times SU(2)}{\mathbb{Z}_2}$ acts on S^3 by

$$g \rightarrow hgh'^{-1}, \quad h, h' \in SU(2). \quad (5)$$

There are several discrete subgroups of $SO(4)$ which act freely on S^3 . Such free actions are given for example by the choices $h \in \mathbb{Z}_p$, $h' \in \mathbb{Z}_q$ where p and q are relatively prime. The quotients of S^3 by the free actions of discrete subgroups of $SO(4)$ are called spherical space forms. For the above example with cyclic groups $\mathbb{Z}_{p,q} := \mathbb{Z}_p \times \mathbb{Z}_q$ the quotients are Lens spaces $L_{p,q}$ [17]. Of these $L_{1,2}$ and $L_{2,1}$ are \mathbb{RP}^3 .

Spherical space forms are prime and admit metrics with constant positive curvature. They have been studied exhaustively from the point of view of quantum gravity by Witt [13].

b) *Hyperbolic spaces*. Consider the hyperboloid

$$\mathcal{H}^+ : \{x = (x_0, \vec{x}) \in \mathbb{R} \times \mathbb{R}^3 \cong \mathbb{R}^4 : (x_0)^2 - (\vec{x})^2 = 1, x_0 > 0\} \quad (6)$$

in \mathbb{R}^4 . The connected Lorentz group \mathcal{L}_+^\uparrow acts transitively on \mathcal{H}^+ . Let $D \subset \mathcal{L}_+^\uparrow$ be a discrete subgroup acting freely on \mathcal{H}^+ . Then \mathcal{H}^+/D is a hyperbolic space.

Hyperbolic spaces are prime and admit metrics with constant negative curvature.

There are other primes as well such as $S^2 \times S^1$ which do not fall into either of these classes.

B. Manifolds with one asymptotic region

These manifolds \mathcal{M}_∞ also have a unique decomposition of the form

$$\mathcal{M}_\infty = \mathbb{R}^d \#_\alpha \mathcal{P}_\alpha \quad (7)$$

where \mathcal{P}_α are the prime manifolds we discussed previously. Manifolds with one asymptotic region can be obtained from closed manifolds \mathcal{M} by removing a point (“point at ∞ ”).

III. ON DIFFEOS

Spatial manifolds of interest for geons are \mathcal{M}_∞ . They serve as Cauchy surfaces in globally hyperbolic spacetimes. Spacetime topology is taken to be $\mathcal{M}_\infty \times \mathbb{R}$ where \mathbb{R} accounts for time.

In the standard Drinfel’d twist approach, the twist F_θ belongs to $\mathbb{C}\mathcal{G} \otimes \mathbb{C}\mathcal{G}$, where \mathcal{G} represents the symmetry group which in relativistic quantum field theory is taken to be \mathcal{P} or its identity component \mathcal{P}_+^\uparrow . In order to let the twist act on a geon spacetime we should identify the substitute for \mathcal{P} or \mathcal{P}_+^\uparrow . To achieve that we need to recall a few properties of quantization of diffeomorphism-invariant theories. We will present a summary of the main ideas here. For a self-contained treatment of the topic we refer the reader to [18].

It is a result of quantization on multiply connected configuration spaces [19] that there is an action on the Hilbert space \mathcal{H} of $\pi_1(\mathcal{Q})$, where \mathcal{Q} is the configuration space of the classical system we want to quantize. This action can also be shown to commute with the action of any observable on \mathcal{H} . Now \mathcal{H} can be decomposed into the direct sum $\mathcal{H} \cong \bigoplus \mathcal{H}^{(l)}$ of carrier spaces of irreducible representation of $\pi_1(\mathcal{Q})$. (More precisely this is so only if $\pi_1(\mathcal{Q})$ is abelian. Otherwise \mathcal{H} carries only the action of the center of the group algebra $\mathbb{C}\pi_1(\mathcal{Q})$, see for instance [18].) Since all the observables commute with the action of $\pi_1(\mathcal{Q})$, they take each $\mathcal{H}^{(l)}$ into itself. These quantizations for different l are generally inequivalent. In other words each $\mathcal{H}^{(l)}$ provides an inequivalent quantization of the classical system [18]. These results have been widely used from molecular physics to quantum field theory. The θ angle of QCD it is in fact understood in such a topological way.

In a theory of quantum gravity we consider $\pi_1(\mathcal{Q})$ as the group to twist. We turn now into the study of what this group looks like.

In general relativity the configuration space is very different from the usual \mathbb{R}^{3n} , as it is in the n -particle

case. Specifically it is constructed from the set of all possible Riemannian metrics on a given space-like Riemannian manifold \mathcal{M} , which we will indicate as $\text{Riem}(\mathcal{M})$. We also require, in order to make sense of concepts constantly used in physics like energy, that \mathcal{M} is asymptotically flat. So we restrict \mathcal{M} to what has been called \mathcal{M}_∞ above. We indicate by $\text{Riem}(\mathcal{M}_\infty)$ the space of metrics on it.

Not all possible metrics on \mathcal{M}_∞ represent physically inequivalent “degrees of freedom” though. Because of diffeomorphism invariance we should consider only $\text{Riem}(\mathcal{M}_\infty)$ upto the action of D^∞ , the diffeos which act trivially at infinity. We thus find for the configuration space \mathcal{Q} of general relativity: $\mathcal{Q} \equiv \text{Riem}(\mathcal{M}_\infty)/D^\infty$.

The next step is to compute the fundamental group of \mathcal{Q} . We first quote the result:

$$\pi_1\left(\text{Riem}(\mathcal{M}_\infty)/D^\infty\right) = D^\infty/D_0^\infty := \text{MCG}(\mathcal{M}_\infty) \quad (8)$$

where D_0^∞ is the (normal) subgroup of D^∞ which is connected to the identity and MCG denotes the Mapping Class Group. This group is an important invariant of topological spaces.

Here is the proof of (8). It can be shown that the action of D^∞ on $\text{Riem}(\mathcal{M}_\infty)$ is free. Thus \mathcal{Q} is the base manifold of a principal bundle $\text{Riem}(\mathcal{M}_\infty)$ with structure group D^∞ . By a well-known theorem of homotopy theory [20], the following sequence of homotopy groups is then exact:

$$\dots \rightarrow \pi_1\left(\text{Riem}(\mathcal{M}_\infty)\right) \rightarrow \pi_1(\mathcal{Q}) \rightarrow \pi_0(D^\infty) \rightarrow \pi_0\left(\text{Riem}(\mathcal{M}_\infty)\right) \rightarrow \dots \quad (9)$$

As the space of Riemmanian asymptotically flat metrics is topologically “trivial”, that is $\pi_n\left(\text{Riem}(\mathcal{M}_\infty)\right) \equiv \mathbb{1}$, $\forall n$, (9) becomes:

$$\mathbb{1} \rightarrow \pi_1(\mathcal{Q}) \rightarrow \pi_0(D^\infty) \rightarrow \mathbb{1} \quad (10)$$

from which (8) follows.

The nontrivial structure of the $\text{MCG}(\mathcal{M}_\infty)$ leads to striking results like the possibility of spinorial states from pure gravity. Let us briefly discuss this interesting result. The group D^∞ contains a diffeo called the 2π -rotation diffeo $R_{2\pi}$. It becomes a 2π rotation on quantum states. It may or may not be an element of D_0^∞ . Now “the momentum constraints” of general relativity imply that D_0^∞ acts as identity on all quantum states. Thus it is only the group D^∞/D_0^∞ (or more generally D/D_0^∞ where D may contain elements which asymptotically act for example as rotations and translations) which can act nontrivially on quantum states. The conclusion in the following relies on this fact.

If $R_{2\pi} \in D_0^\infty$ then it maps to the identity in D^∞/D_0^∞ and on quantum states.

If $R_{2\pi} \notin D_0^\infty$, then it does not map to identity in D^∞/D_0^∞ and can act nontrivially on quantum states.

For $d \geq 3$, $R_{2\pi}^2$ is always in D_0^∞ and hence always acts trivially on quantum states.

Thus if $d \geq 3$ and $R_{2\pi} \notin D_0^\infty$, there can exist quantum geons with 2π rotation $= -\mathbb{1}$ on their Hilbert space.

In fact suppose that $\psi \in \mathcal{H}$ is a physical state on which $R_{2\pi}$ does not act trivially, $\hat{R}_{2\pi}\psi \neq \psi$. But $R_{4\pi} = R_{2\pi}^2$ acts trivially on \mathcal{H} . Then the state $\psi' := (\frac{\mathbb{1} - \hat{R}_{2\pi}}{2})\psi$ is spinorial:

$$\hat{R}_{2\pi}\psi' = \frac{1}{2}(\hat{R}_{2\pi} - \hat{R}_{4\pi})\psi = -\psi' \quad . \quad (11)$$

It was a remarkable observation of Friedman and Sorkin [10–12] that there exist primes \mathcal{P}_α such that $R_{2\pi} \notin D_0^\infty$ for $\mathcal{M} = \mathbb{R}^d \# \mathcal{P}_\alpha$. These are the “spinorial” primes. The quantisation of the metric of such $\mathbb{R}^3 \# \mathcal{P}_\alpha$ can lead to vector states with spin $\frac{1}{2} + n$ ($n \in \mathbb{Z}^+$). Thus we can have “spin $\frac{1}{2}$ from gravity”.

For $d = 2$, the situation is similar, but $R_{2\pi}^2$ or any nontrivial power of $R_{2\pi}$, need not be in D_0^∞ . That is indeed the case for $\mathbb{R}^2 \# T^2$ [21]. That means that the quantum states for such geon manifolds can have fractional spin, can be anyons.

Notation.

Here we introduce some notation. We will call the diffeo groups of $\mathcal{M}_\infty = \mathbb{R}^d \# \mathcal{P}_\alpha$ which are asymptotically Poincaré, asymptotically identity and the component connected to the identity of the latter as $D^{(1)}$, $D^{(1)\infty}$ and $D_0^{(1)\infty}$ respectively. We will also refer to $D^{(1)\infty}/D_0^{(1)\infty}$ as the internal diffeos of the prime \mathcal{P}_α . Similarly $D^{(N)}$, $D^{(N)\infty}$ and $D_0^{(N)\infty}$ will refer to the corresponding groups in the case of N -geon manifolds $\mathcal{M}_\infty = \mathbb{R}^d \# \mathcal{P}_\alpha \# \dots \# \mathcal{P}_\alpha$, where the primes are all the same. They are appropriate for constructing vector states of several identical geons.

The MCG of an N -geon manifold can be decomposed into semi-direct products involving three groups:

$$D^{(N)\infty}/D_0^{(N)\infty} \equiv \left(\mathcal{S} \rtimes \left[\times^N D^{(1)\infty}/D_0^{(1)\infty} \right] \right) \rtimes S_N \quad . \quad (12)$$

Here $A \rtimes B$ indicates the semi-direct product of A with B where A is the normal subgroup.

In the above we could remove the brackets as it has been shown in [22, 23] that the above semi-direct

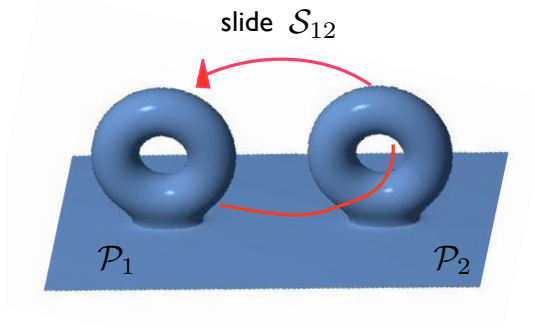


FIG. 3. $\mathbb{R}^d \# \mathcal{P}_1 \# \mathcal{P}_2$

product is associative.

The last two factors in (12) are easily understood, the second term being the N -th direct product of the MCG of the single geon manifold $\mathcal{M}_\infty = \mathbb{R}^d \# \mathcal{P}_\alpha$ and S_N being the usual permutation group of N elements that consists of elements which permute the geons.

The first term, namely \mathcal{S} , is called the group of “slides” and consists of diffeos which take one prime through another along non-contractible loops. The existence of such a term is strictly linked with the fact that the primes are not simply connected. In fact elements of \mathcal{S} can be described using elements of fundamental groups of the single primes \mathcal{P}_α . Since we are not interested in the full details of the MCG, we refer the reader to the literature for further details [22, 23] while we now move on to the analysis of the $N = 2$ case where we can also get a better understanding of what slides represent.

As we said the group $D^{(2)\infty}/D_0^{(2)\infty}$ of the manifold $\mathbb{R}^d \# \mathcal{P}_\alpha \# \mathcal{P}_\alpha$ appropriate for two identical geons contains diffeos corresponding to the exchange $E^{(2)}$ of geons and a new type of diffeos called slides besides the diffeos $D^{(1)\infty}/D_0^{(1)\infty}$ of $\mathbb{R}^d \# \mathcal{P}_\alpha$.

If $\mathbb{R}^d \# \mathcal{P}_\alpha \# \mathcal{P}_\alpha$ is represented as in Fig. 3 with bumps representing \mathcal{P}_α , the exchange diffeo $E^{(2)}$ can be regarded as moving the geons so that they exchange places. This diffeo (mod $D_0^{(2)\infty}$) is the generator of S_2 in (12). For $d = 3$, $E^{(2)2} \in D_0^{(2)\infty}$, but for $d = 2$ that is not so. Thus for $d = 2$, we can have geons with fractional statistics [21].

Slides $\mathcal{S}^{(2)}$ arise because for $\mathcal{P}_\alpha \neq S^d$, $\pi_1(\mathcal{P}_\alpha) \neq \{e\}$ for $d = 2$ (where $\mathcal{P}_\alpha = T^2$), and $d = 3$ (in view of the now-proved Poincaré conjecture). Thus let L be a non-contractable loop threading $\mathbb{R}^d \# \mathcal{P}_\alpha^{(1)}$, where $\mathcal{P}_\alpha^{(j)}$ are primes and let \mathcal{B}_p be a ball containing a point P on L in its interior. Then $\mathcal{S}_{21}^{(2)}$, the slide of $\mathcal{P}_\alpha^{(2)}$ along L

through $\mathcal{P}_\alpha^{(1)}$, is obtained by attaching $\mathcal{P}_\alpha^{(2)}$ to $\partial\mathcal{B}_p$ and dragging it along L by moving p in a loop around L . Note that the slide $\mathcal{S}_{12}^{(2)}$ of $\mathcal{P}_\alpha^{(1)}$ through $\mathcal{P}_\alpha^{(2)}$ is not equal to $\mathcal{S}_{21}^{(2)}$.

IV. QUANTUM FIELDS

In standard quantum physics, there is a relation between spacetime symmetries like the Poincaré group \mathcal{P}_+^\uparrow and the statistics group that implements the identity of particles. It can be described as follows. An element α of the Poincaré group acts on a member ψ of the single particle Hilbert space \mathcal{H} by pullback:

$$(\alpha\psi)(x) = \psi(\alpha^{-1}x) \quad . \quad (13)$$

This action extends to the N -particle Hilbert space $\mathcal{H}^{\otimes N}$ via the coproduct Δ_0 :

$$\Delta_0(\alpha) = \alpha \otimes \alpha \quad (14)$$

Thus on $\mathcal{H}^{\otimes N}$, it acts by

$$\underbrace{(\mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes \Delta_0)}_{(N-1) \text{ factors}} \underbrace{(\mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes \Delta_0)}_{(N-2) \text{ factors}} \dots \Delta_0(\alpha) = \underbrace{\alpha \otimes \alpha \otimes \dots \otimes \alpha}_{N \text{ factors}} \quad (15)$$

The statistics group expressing the identity of particles must commute with the action of the symmetry group. This requirement just says that symmetry transformations, such as Lorentz transformations, should not spoil particle identity. It is fulfilled by the permutation group S_N which permutes the factors in the tensor product

$$\psi_1 \otimes \psi_2 \otimes \dots \otimes \psi_N \in \mathcal{H}^{\otimes N} \quad (16)$$

Quantum fields compatible with the symmetry group such as \mathcal{P}_+^\uparrow and implementing statistics exist. For these fields, the permutation group S_N and say the Poincaré transformation commute when acting on N -particle in- or out- states.

In the case of geon spacetimes the situation is more involved. In fact while for the standard coproduct like that in (14), the internal symmetry $D^{(1)\infty}/D_0^{(1)\infty}$ acts by its diagonal map into $D^{(N)\infty}/D_0^{(N)\infty}$ and that action commutes with S_N , the slides present a more complex story. They do not commute with S_N (nor

with $\alpha \otimes \dots \otimes \alpha$ for $\alpha \in D^{(1)\infty}/D_0^{(1)\infty}$) and can change representations of S_N : they can convert bosons into fermions! For such reasons, Sorkin and Surya have suggested that elements of \mathcal{S} represent interactions of geons. Elsewhere we will implement this idea in a qft. But for now we let \mathcal{S} act by the identity representation on quantum states. That means that we will work with $\left[\times^N D^{(1)\infty}/D_0^{(1)\infty}\right] \rtimes S_N$ and their group algebra.

The generalisations of symmetry groups are Hopf algebras. This more general setting is needed by us below where we will work with the group algebra $\mathbb{C}(D^{(1)\infty}/D_0^{(1)\infty})$ with a twisted coproduct. In that case too, the algebra defining statistics is in the commutant of the coproduct. It is still S_N , but acts differently on $\mathcal{H}^{\otimes N}$.

From (12), slides form an invariant subgroup in $D^{(N)\infty}/D_0^{(N)\infty}$. For this reason, slides can be represented by identity on quantum states. Sorkin and Surya [22, 23] have suggested that we do so motivated by the considerations above. We follow their suggestion.

V. TWISTS OF GEON SPACETIMES: THE CHOICE

The Drinfel'd twist F_θ of the Poincaré group algebra $\mathbb{C}\mathcal{P}_+^\uparrow$, which let the latter act on the Moyal plane $\mathcal{A}_\theta(\mathbb{R}^d)$, is by now well-known [24]. If P_μ are the translations generators of \mathcal{P}_+^\uparrow and $\theta = [\theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R}]$ is the matrix characterising the Moyal \star -product,

$$f \star g = f e^{-\frac{i}{2} \overleftarrow{\mathcal{P}}_\mu \theta^{\mu\nu} \overrightarrow{\mathcal{P}}_\nu} g, \quad f, g \in \mathcal{A}_\theta(\mathbb{R}^d), \quad (17)$$

where $\mathcal{P}_\mu = -i\partial_\mu$ is the representative of P_μ on $\mathcal{A}_\theta(\mathbb{R}^d)$, then

$$F_\theta = e^{-\frac{i}{2} P_\mu \otimes \theta^{\mu\nu} P_\nu}. \quad (18)$$

Its realization on the algebra of functions is just the term appearing in between the two functions f and g in (17). We will indicate it by a script \mathcal{F}_θ :

$$\mathcal{F}_\theta = e^{\frac{i}{2} \overleftarrow{\partial}_\mu \theta_{\mu\nu} \overrightarrow{\partial}_\nu}. \quad (19)$$

The twisted coproduct Δ_θ of $\mathbb{C}\mathcal{P}_+^\uparrow$ is fixed by those of the elements $g \in \mathcal{P}_+^\uparrow$:

$$\Delta_\theta(g) = F_\theta^{-1}(g \otimes g)F_\theta. \quad (20)$$

We want to generalise F_θ to geonic diffeos and in particular to $D^{(1)\infty}/D_0^{(1)\infty}$. The motivation is as follows.

If a sphere S^{d-1} encloses the prime in $\mathbb{R}^d \# \mathcal{P}_\alpha$ in the sense that the complement of this sphere in $\mathbb{R}^d \# \mathcal{P}_\alpha$ is homeomorphic to $\mathbb{R}^d / \mathcal{B}^d$ where \mathcal{B}^d is the d -dimensional ball, then by suitably adjoining elements of D_0^∞ , we can ensure that D^∞ / D_0^∞ acts as the identity outside S^{d-1} . So these diffeos can be taken to be localised on the geon. If the geon size is of the order of the Planck volume, the action of $D^{(1)\infty} / D_0^{(1)\infty}$ is also confined to such Planck volumes (It is possible to define geon sizes in a diffeo-invariant way [21]). As explained in the introduction, at these scales we expect the spacetime to be noncommutative and the action of the symmetry group to be consequently twisted.

We will generalise F_θ to $D^{(1)\infty} / D_0^{(1)\infty}$ and after that twist using elements of $D^{(1)\infty} / D_0^{(1)\infty}$. Then, as we shall see, spacetimes become noncommutative on the above Planck-scale volumes. This is in accordance with the arguments of DFR [1].

Thus the choice of twists using $D^{(1)\infty} / D_0^{(1)\infty}$ appears to be one good way to implement the DFR ideas.

It is also one way to incorporate aspects of the topology of geons in these basic quantum field theories as we shall see.

VI. TWISTS OF GEON SPACETIMES: COASSOCIATIVE COPRODUCTS

The generalisation of F_θ to $D^{(1)\infty} / D_0^{(1)\infty}$ is not immediate since $D^{(1)\infty} / D_0^{(1)\infty}$ is discrete. It can be finite or infinite, but it is certainly discrete. So we must know how to adapt F_θ to discrete groups. The difficulty comes from the fact that for Lie groups, we write F_θ in terms of the exponential of the tensor product of Lie algebra elements, as in (18). There is no analogue of the Lie algebra for discrete groups. As we will shortly see, writing the twist F_θ in momentum space sheds light on the path to follow for the generalization.

The plane waves e_p , $e_p(x) = e^{ip \cdot x}$, carry the irreducible representations of the translation subgroup of $\mathbb{C}\mathcal{P}_+^\uparrow$. Since

$$\mathcal{P}_\mu e_p = p_\mu e_p \quad , \quad (21)$$

the restriction of \mathcal{F}_θ (19) to $e_p \otimes e_q$ is given by

$$\mathcal{F}_\theta e_p \otimes e_q = e^{-\frac{i}{2} p_\mu \theta^{\mu\nu} q_\nu} e_p \otimes e_q \quad . \quad (22)$$

Let \mathfrak{P}_p be the projection operator which acting on functions of \mathbb{R}^d projects to the subspace spanned by

e_p . It is thus the projector to the irreducible representation of the translation subgroup identified by the real vector “ p ”. For a particle of mass m , for which $p_0 = \sqrt{\vec{p}^2 + m^2}$, we can define \mathfrak{P}_p by requiring that

$$\mathfrak{P}_p e_q = 2|p_0| \delta^{(3)}(\vec{p} - \vec{q}) e_p \quad . \quad (23)$$

Then we can see that

$$\mathcal{F}_\theta = \int d\mu(p) d\mu(q) e^{-\frac{i}{2} p \wedge q} \mathfrak{P}_p \otimes \mathfrak{P}_q, \quad d\mu(p) := \frac{d^3 p}{2\sqrt{\vec{p}^2 + m^2}} \quad (24)$$

where $p \wedge q := p_\mu \theta_{\mu\nu} q_\nu$, and that

$$F_\theta = \int d\mu(p) d\mu(q) e^{-\frac{i}{2} p \wedge q} \mathfrak{P}_p \otimes \mathfrak{P}_q \quad . \quad (25)$$

If e_p is off-shell, so that p_0 is not constrained to be $\sqrt{\vec{p}^2 + m^2}$, we can still write \mathcal{F}_θ in terms of projections by slightly modifying (24).

A. A Simple Generalisation to Discrete Abelian Groups

It is possible to find a simple generalisation of (24-25) to discrete abelian groups. We first discuss this generalisation.

Consider first the group

$$\mathbb{Z}_n = \{\xi^k \equiv e^{i\frac{2\pi}{n}k} : k = 0, 1, \dots, (n-1)\} \quad . \quad (26)$$

Its IRR’s ϱ_m are all one-dimensional and given by its characters χ_m :

$$\chi_m(\xi) = \xi^m, \quad m \in \{0, 1, \dots, (n-1)\} \quad . \quad (27)$$

Then if $\hat{\xi}$ is the operator representing ξ on the space on which it acts, the projector \mathfrak{P}_m to the IRR ϱ_m is

$$\mathfrak{P}_m = \frac{1}{n} \sum_{k=0}^{n-1} \bar{\chi}_m(\xi^k) \hat{\xi}^k \quad (28)$$

This follows from

$$\xi^l \mathfrak{P}_m = \frac{1}{n} \sum_{k=0}^{n-1} \bar{\chi}_m(\xi^k) \xi^{k+l} = \frac{1}{n} \sum_{k=l}^{n+l-1} \bar{\chi}_m(\xi^{k-l}) \xi^k = \chi_m(\xi^l) \mathfrak{P}_m \quad (29)$$

where we used the fact that

$$\bar{\chi}(\xi^l) \chi(\xi^l) = 1, \quad \bar{\chi}(\xi^l) = \chi(\xi^{-l}), \quad (30)$$

and the orthogonality relations,

$$\frac{1}{n} \sum_{\xi} \bar{\chi}_m(\xi) \chi_n(\xi) = \delta_{m,n}, \quad (31)$$

that imply,

$$\mathfrak{P}_m \mathfrak{P}_n = \delta_{m,n} \mathfrak{P}_n. \quad (32)$$

Note that \mathfrak{P}_m is the image of

$$\mathbb{P}_m = \frac{1}{n} \sum_{k=0}^{n-1} \bar{\chi}_m(\xi^k) \xi^k \quad (33)$$

in the group algebra $\mathbb{C}\mathbb{Z}_n$ and that

$$\mathbb{P}_m \mathbb{P}_n = \delta_{m,n} \mathbb{P}_n, \quad \sum_{m=0}^{n-1} \mathbb{P}_m = \mathbb{1}. \quad (34)$$

B. The case of $D^{(1)\infty}/D_0^{(1)\infty}$

From $D^{(1)\infty}/D_0^{(1)\infty}$, we pick its maximal abelian subgroup A and assume for the moment that A is finite. Then A is the direct product of cyclic groups:

$$A = \mathbb{Z}_n \times \mathbb{Z}_{n_2} \times \dots \times \mathbb{Z}_{n_k}. \quad (35)$$

Its IRR's are given by:

$$\varrho_{m_1} \otimes \varrho_{m_2} \otimes \dots \otimes \varrho_{m_k}, \quad m_j \in \{0, 1, \dots, n_j - 1\}, \quad (36)$$

with characters

$$\chi_{\vec{m}} = \prod_i \chi_{m_i} \quad (37)$$

and projectors $\mathfrak{P}_{\vec{m}} = \otimes_i \mathfrak{P}_{m_i}$ on the representation space or projectors

$$\mathbb{P}_{\vec{m}} = \otimes_i \mathbb{P}_{m_i}, \quad \mathbb{P}_{\vec{m}} \mathbb{P}_{\vec{m}'} = \delta_{\vec{m}, \vec{m}'} \mathbb{P}_{\vec{m}}, \quad \sum_{\vec{m}} \mathbb{P}_{\vec{m}} = \text{identity of } A \quad (38)$$

in the group algebra $\mathbb{C}A$. (The summation of m_j in (38) is from 0 to $n_j - 1$).

Let $\theta = [\theta_{ij} = -\theta_{ji} \in \mathbb{R}]$ be an antisymmetric matrix with constant entries. Following (25), we can write a Drinfel'd twist using elements of $\mathbb{C}A$:

$$F_\theta = \sum_{\vec{m}, \vec{m}'} e^{-\frac{i}{2} m_i \theta_{ij} m'_j} \mathbb{P}_{\vec{m}} \otimes \mathbb{P}_{\vec{m}'} \quad . \quad (39)$$

But there are quantisation conditions on θ_{ij} . That is because ϱ_m and ϱ_{m+n} give the same IRR for \mathbb{Z}_n as (27) shows. That means that \vec{m} and $\vec{m} + (0, \dots, 0, n_i, 0, \dots, 0)$ give the same IRR $\varrho_{\vec{m}}$, n_i being the i^{th} entry. Since F_θ must be invariant under these shifts, we find that θ_{ij} is restricted to the values

$$\theta_{ij} = \frac{4\pi}{n_{ij}} \quad (40)$$

where

$$\frac{n_i}{n_{ij}}, \frac{n_j}{n_{ij}} \in \mathbb{Z} \quad . \quad (41)$$

The twist (20) of the canonical coproduct of \mathcal{P}_+^\uparrow using F_θ leads to a coassociative coproduct. Similarly the twist of the coproduct of $D^{(1)\infty}/D_0^{(1)\infty}$ or any of its subgroups leads to a coassociative coproduct. That is because the twist involves the abelian algebra $\mathbb{C}A$. As we will further discuss later on, the spacetime algebra is associative, but not commutative if a $\theta_{ij} = -\theta_{ji} \neq 0$ [25].

Remarks

- a) The condition (41) has a solution $n_{ij} \neq \pm 1$ only if n_i and n_j have a common factor ($\neq \pm 1$). Thus if say $n_i = 2, n_j = 3$ for some i, j then $n_{ij} = \pm 1$. For either of these solutions,

$$e^{-\frac{i}{2} m_i \theta_{ij} m'_j} = 1 \quad \text{or} \quad \theta_{ij} \text{ is effectively equivalent to } 0 \quad . \quad (42)$$

- b) There are many instances where A contains factors of \mathbb{Z} . The IRR's ϱ_φ of \mathbb{Z} are given by points of

$$S^1 = \{e^{i2\pi\varphi} : 0 \leq \varphi \leq 1\}:$$

$$\varrho_\varphi : n \in \mathbb{Z} \rightarrow e^{i2\pi n\varphi}. \quad (43)$$

Note that

$$\varrho_\varphi = \varrho_{\varphi+1} \quad . \quad (44)$$

Suppose now that $A = \times_{i=1}^k \mathbb{Z}_{n_i} \times \mathbb{Z}$. Now its IRR's are labelled by the vector $(\vec{m}, \varphi) = (m_1, \dots, m_k, \varphi)$.

The twist F_θ is written as

$$\begin{aligned} F_\theta = \sum_{\vec{m}, \vec{m}'} \int_0^1 d\varphi \int_0^1 d\varphi' e^{-\frac{i}{2} m_i \theta_{ij} m'_j} \times \\ \times e^{-\frac{i}{2} [m_i (\theta_{m_i, k+1}) \varphi' - \varphi (\theta_{m_i, k+1}) m'_i]} \mathbb{P}_{(\vec{m}, \varphi)} \otimes \mathbb{P}_{(\vec{m}', \varphi')} \end{aligned} \quad (45)$$

But the periodicity in φ, φ' is 1 and hence $\theta_{m_i, k+1} = \pm 4\pi$ and the second exponential in (45) is $\mathbb{1} \otimes \mathbb{1}$. In short, F_θ has no twist factor involving \mathbb{Z} and F_θ reduces back to the earlier expression (39). If there are say *two factors* of \mathbb{Z} so that $A = \times_{i=1}^{k-1} \mathbb{Z}_{n_i} \otimes \mathbb{Z} \otimes \mathbb{Z}$ the second exponential in (45) is replaced by

$$e^{\varphi \theta_{k, k+1} \varphi'} \quad (46)$$

and we require its periodicity in φ and φ' . Hence $\theta_{k, k+1} \simeq 0$. In this way, we see that F_θ depends nontrivially only on compact abelian discrete groups.

- c) Later in section IX, we will argue that the twists found above seem general so long as we insist on the coassociativity of the coproduct (or equivalently the associativity of the spacetime algebra).

VII. ON TWISTED SYMMETRISATION AND ANTISYMMETRISATION

Let \mathcal{H} be a one-geon Hilbert space. It carries a representation of $D^{(1)}$ or more generally of D . The “momentum constraint” is implemented by requiring that $D_0^\infty \rightarrow \mathbb{1}$ in this representation which we assume is satisfied.

Let τ_0 be the flip operator on $\mathcal{H} \otimes \mathcal{H}$:

$$\tau_0 \alpha \otimes \beta = \beta \otimes \alpha, \quad \alpha, \beta \in \mathcal{H} \quad . \quad (47)$$

When the coproduct is Δ_0 , $\Delta_0(d) = d \otimes d$ for $d \in D$, τ_0 commutes with Δ_0 (by d here we mean the representation of d on \mathcal{H}). So the subspaces $\frac{\mathbb{1} \pm \tau_0}{2} \mathcal{H} \otimes \mathcal{H}$ are invariant under diffeos and carry the identity representation of D_0^∞ . We can then use them to define bosonic and fermionic geons.

But if we deform Δ_0 into (20), τ_0 does not commute with $\Delta_\theta(d)$ for all d anymore if $F_\theta \neq \mathbb{1} \otimes \mathbb{1}$. So the subspaces $\frac{\mathbb{1} \pm \tau_0}{2} \mathcal{H} \otimes \mathcal{H}$ are not diffeomorphism invariant, nor need they fulfill the constraint $\Delta_\theta(d) [\tau_0(\alpha \otimes \beta)] = \tau_0 [\Delta_\theta(d)(\alpha \otimes \beta)]$ for $d \in D^{(1)}$. That means that bosons and fermions cannot be associated with the subspaces $\frac{\mathbb{1} \pm \tau_0}{2} \mathcal{H} \otimes \mathcal{H}$.

Instead, as discussed elsewhere [26, 27], one should use the twisted flip operator

$$\tau_\theta = F_\theta^{-1} \tau_0 F_\theta, \quad \tau_\theta^2 = \mathbb{1} \otimes \mathbb{1} \quad (48)$$

which commutes with the twisted coproduct $\Delta_\theta(d)$. Bosonic and fermionic geons are thus associated with the subspaces $\frac{\mathbb{1} \pm \tau_\theta}{2} \mathcal{H} \otimes \mathcal{H}$.

The twist depends on D^∞/D_0^∞ . So these twisted subspaces incorporate at least aspects of the internal diffeos of geons unlike τ_0 . Such a twist of flip is a consequence of deforming the coproduct to Δ_θ . As we will discuss, this deformation introduces spacetime noncommutativity localised at the geon. Further there are outlines available for an approach to build an orderly quantum field theory (compatibly with the DFR suggestion) incorporating this noncommutativity and deformed statistics, and transforming by the twisted coproduct. These are all attractive aspects of introducing the twist F_θ .

VIII. COVARIANT QUANTUM FIELDS: THE MOYAL PLANE

Elsewhere [14], we have carefully discussed the notion of covariant fields in general including in particular the Moyal plane $\mathcal{A}_\theta(\mathbb{R}^d)$. This concept in the limit $\theta \rightarrow 0$ reduces to the corresponding well-known concept for $\theta = 0$. We give a short review of our earlier work [14] for the Moyal plane and then adapt it to discrete Hopf algebras

A. The $\theta = 0$, Commutative Plane $\mathcal{A}_0(\mathbb{R}^d)$

Consider a scalar quantum field φ_0 based on the commutative spacetime algebra $\mathcal{A}_0(\mathbb{R}^d)$. φ_0 depends on the spacetime point x while $\varphi_0(x)$ is an “operator” on a Hilbert space. Each element g of an appropriate diffeo group acts on x ,

$$g : x \rightarrow gx \quad (49)$$

and on fields on \mathbb{R}^d by pull-back:

$$\varphi_0 \rightarrow g\varphi_0, \quad (g\varphi_0)(x) = \varphi_0(g^{-1}x) \quad . \quad (50)$$

It acts on $\varphi_0(x)$ also by a unitary operator $U(g)$:

$$\varphi_0(x) \rightarrow U(g)\varphi_0(x)U(g)^{-1} \quad . \quad (51)$$

The field is said to be *covariant* if it is invariant under the combined action of both:

$$U(g)\varphi_0(g^{-1}x)U(g)^{-1} = \varphi_0(x) \quad . \quad (52)$$

In the literature, this equation is often written as

$$U(g)\varphi_0(x)U(g)^{-1} = \varphi_0(gx) \quad . \quad (53)$$

Poincaré invariant qft's are based on covariant fields. Covariance ensures that the spacetime action of the Poincaré group is implementable by unitary operators. Its classical version would ensure that spacetime transformations are canonically implemented.

A Poincaré-invariant qft has a unique vacuum $|0\rangle$ invariant under $U(g) : U(g)|0\rangle = |0\rangle$. It is a cyclic vector. Using this fact, we can rewrite the covariance of fields in the following way:

$$U(g)\varphi_0(g^{-1}x_1)\varphi_0(g^{-1}x_2)\dots\varphi_0(g^{-1}x_n)|0\rangle = \varphi_0(x_1)\varphi_0(x_2)\dots\varphi_0(x_n)|0\rangle \quad . \quad (54)$$

But note that this uses the canonical coproduct $\Delta_0(g)$ when transforming the spacetime arguments.

The implications of this equation become transparent if written in terms of in (or out) fields. The in field

φ_0^{in} of mass m has the expansion

$$\varphi_0^{\text{in}} = \int d\mu(p) \left[c_p^{\text{in}\dagger} e_p + h.c. \right], \quad e_p = e^{-ip \cdot x}, \quad d\mu(p) = \frac{d^3 p}{2\sqrt{p^2 + m^2}} \quad . \quad (55)$$

We consider the implications of (54) using (55) for translations and Lorentz transformations in turn.

1. Translations

Let first check for covariance under translation.

If P_μ are the translation generators on the Hilbert space with

$$[P_\mu, c_p^{\text{in}\dagger}] = p_\mu c_p^{\text{in}\dagger} \quad (56)$$

and $\mathcal{P}_\mu = -i\partial_\mu$, the representative of translation generators on spacetime, we have

$$[P_\mu, c_p^{\text{in}\dagger}] e_p + c_p^{\text{in}\dagger} (\mathcal{P}_\mu e_p) = 0 \quad . \quad (57)$$

So φ_0 has translational covariance. Notice that we can deduce (56) from (57) and $P_\mu e_p = -p_\mu e_p$.

We can now check (54) for multi-particle states in terms of creation-annihilation operators. The coproduct $\Delta_0(g) = g \otimes g$ gives

$$\Delta_0(\mathcal{P}_\mu) = \mathcal{P}_\mu \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{P}_\mu \quad . \quad (58)$$

From this follows that (54) is satisfied at the two-particle level as well. Explicitly,

$$\begin{aligned} & \int \prod_i d\mu(p_i) \left[P_\mu, c_{p_1}^{\text{in}\dagger} c_{p_2}^{\text{in}\dagger} \right] |0\rangle e_{p_1} \otimes e_{p_2} + \\ & \quad + \int \prod_i d\mu(p_i) c_{p_1}^{\text{in}\dagger} c_{p_2}^{\text{in}\dagger} |0\rangle \Delta_0(\mathcal{P}_\mu) e_{p_1} \otimes e_{p_2} = \\ & = \int \prod_i d\mu(p_i) \left\{ \left(\left[P_\mu, c_{p_1}^{\text{in}\dagger} \right] c_{p_2}^{\text{in}\dagger} + c_{p_1}^{\text{in}\dagger} \left[P_\mu, c_{p_2}^{\text{in}\dagger} \right] \right) |0\rangle e_{p_1} \otimes e_{p_2} + \right. \\ & \quad \left. + c_{p_1}^{\text{in}\dagger} c_{p_2}^{\text{in}\dagger} |0\rangle \left[\mathcal{P}_\mu \otimes \mathbb{1} + \mathbb{1} \otimes \mathcal{P}_\mu \right] e_{p_1} \otimes e_{p_2} \right\} = 0 \quad . \quad (59) \end{aligned}$$

From this calculation, it is evident that (54) generalises to generic N :

$$\begin{aligned} & \int \prod_i d\mu(p_i) \left[P_\mu, c_{p_1}^{\text{in}\dagger} \dots c_{p_N}^{\text{in}\dagger} \right] |0\rangle e_{p_1} \otimes \dots \otimes e_{p_N} + \\ & + \int \prod_i d\mu(p_i) c_{p_1}^{\text{in}\dagger} \dots c_{p_N}^{\text{in}\dagger} |0\rangle \left[\mathbb{1} \otimes \mathbb{1} \otimes \dots \otimes \Delta_0 \right] \dots \left[\mathbb{1} \otimes \Delta_0 \right] \Delta_0(\mathcal{P}_\mu) e_{p_1} \otimes \dots \otimes e_{p_N} . \end{aligned} \quad (60)$$

Thus (57) is necessary for covariance.

The consistency of the adjoint of (56),

$$[P_\mu, c_p^{\text{in}}] = -p_\mu c_p^{\text{in}} \quad (61)$$

with covariance can also be established in the same way by acting with fields on bra vectors.

2. Lorentz Transformations

Next we consider covariance under Lorentz transformations $\Lambda \in \mathcal{P}_+^\uparrow$ and the transformation rules it implies on $c_p^{\text{in}\dagger}$, c_p^{in} . Under Λ ,

$$\Lambda : e_p \rightarrow e_{\Lambda p} \quad (62)$$

while $d\mu(\Lambda p) = d\mu(p)$. Thus the equation

$$U(\Lambda) c_p^{\text{in}\dagger} U(\Lambda)^{-1} = c_{\Lambda p}^{\text{in}\dagger} \quad (63)$$

is immediate, while its adjoint is

$$U(\Lambda) c_p^{\text{in}} U(\Lambda)^{-1} = c_{\Lambda p}^{\text{in}} \quad (64)$$

As before it generalises to products of N c_p^{in} 's.

B. The Moyal Plane $\mathcal{A}_\theta(\mathbb{R}^d)$

Let us next review the Moyal case. For single particle states, the twisted and untwisted actions do not differ, so we consider the two-particle sector.

The twisted in fields will be written as

$$\varphi_\theta^{\text{in}} = \int d\mu(p) \left[a_p^{\text{in}\dagger} e_p + h.c. \right] \quad (65)$$

where the relation of $a_p^{\text{in}\dagger}$, a_p^{in} to $c_p^{\text{in}\dagger}$, c_p^{in} will be determined from covariance.

As usual we assume that there is a unique Poincaré-invariant vacuum annihilated by a_p^{in} .

The unitary operators representing $g \in \mathcal{P}_+^\uparrow$ on the Hilbert space of states will be denoted by $U_\theta(g)$ with $U_0(g) = U(g)$. We will show that $U_\theta(g)$ has the same expression as $U(g)$ in terms of $c_p^{\text{in}\dagger}$, c_p^{in} so that we will later write

$$U_\theta(g) = U(g). \quad (66)$$

Consider

$$\int d\mu(p_1) d\mu(p_2) a_{p_1}^{\text{in}\dagger} a_{p_2}^{\text{in}\dagger} |0\rangle e_{p_1} \otimes e_{p_2}. \quad (67)$$

The twisted coproduct is¹

$$\Delta_\theta(g) = F_\theta^{-1}(g \otimes g) F_\theta, \quad F_\theta = e^{-\frac{i}{2} P \wedge P} \rightarrow \mathcal{F}_\theta = e^{\frac{i}{2} \overleftarrow{\partial} \wedge \overrightarrow{\partial}} \text{ on } e_{p_1} \otimes e_{p_2}. \quad (68)$$

As

$$\Delta_\theta(P_\mu) = \Delta_0(P_\mu) = P_\mu \otimes \mathbb{1} + \mathbb{1} \otimes P_\mu, \quad (69)$$

we find as before that

$$[P_\mu, a_p^{\text{in}\dagger}] = p_\mu a_p^{\text{in}\dagger}, \quad [P_\mu, a_p^{\text{in}}] = -p_\mu a_p^{\text{in}}. \quad (70)$$

As for Lorentz transformations,

$$\Delta_\theta(\Lambda) e_{p_1} \otimes e_{p_2} = e^{\frac{i}{2}(\Lambda p_1) \wedge (\Lambda p_2)} e^{-\frac{i}{2} p_1 \wedge p_2} e_{\Lambda p_1} \otimes e_{\Lambda p_2}. \quad (71)$$

¹ We want to explain our notation to the reader here. Previously we already introduced the “definition” of the wedge symbol among four-vector as

$$p_1 \wedge p_2 := p_{1\mu} \theta^{\mu\nu} p_{1\nu}.$$

(This differs from the wedge symbol among differential forms.) In the following we extend this definition to operators as well. In the case of a bi-operator, a tensor product is to be understood as implicit when a wedge symbol appears,

$$P \wedge P := P_\mu \theta^{\mu\nu} \otimes P_\nu$$

whereas there is no tensor product symbol in the case when one of the p 's is a vector:

$$p \wedge P := \overline{22} p_\mu \theta^{\mu\nu} P_\nu.$$

These notations should not cause confusion.

Covariance hence requires that

$$\begin{aligned} U_\theta(\Lambda) \int d\mu(p_1) d\mu(p_2) a_{p_1}^{\text{in}\dagger} a_{p_2}^{\text{in}\dagger} |0\rangle e^{\frac{i}{2}(\Lambda p_1) \wedge (\Lambda p_2)} e^{-\frac{i}{2} p_1 \wedge p_2} = \\ = \int d\mu(p_1) d\mu(p_2) a_{p_1}^{\text{in}\dagger} a_{p_2}^{\text{in}\dagger} |0\rangle. \end{aligned} \quad (72)$$

A solution of this equation is

$$a_p^{\text{in}\dagger} = c_p^{\text{in}\dagger} e^{\frac{i}{2} p \wedge P}, \quad U_\theta(\Lambda) = U_0(\Lambda) \quad . \quad (73)$$

We will argue below that this solution is unique.

At the level of wave functions, compatibility with the coproduct Δ_θ requires that we twist-symmetrise using

$$\tau_\theta = F_\theta^{-1} \tau_0 F_\theta, \quad \tau_0 \alpha \otimes \beta = \beta \otimes \alpha \quad . \quad (74)$$

If \otimes_{S_θ} denotes twisted Bose-symmetrised tensor product,

$$e_{p_1} \otimes_{S_\theta} e_{p_2} = \frac{1 + \tau_\theta}{2} e_{p_1} \otimes e_{p_2} \quad , \quad (75)$$

then

$$e_{p_1} \otimes_{S_\theta} e_{p_2} = \frac{1}{2} \left[e_{p_1} \otimes e_{p_2} + e^{ip_1 \wedge p_2} e_{p_2} \otimes e_{p_1} \right] \quad , \quad (76)$$

and

$$e_{p_2} \otimes_{S_\theta} e_{p_1} = \frac{1}{2} \left[e_{p_2} \otimes e_{p_1} + e^{-ip_1 \wedge p_2} e_{p_1} \otimes e_{p_2} \right] = e^{-ip_1 \wedge p_2} e_{p_1} \otimes_{S_\theta} e_{p_2} \quad . \quad (77)$$

Thus the equality of

$$\int d\mu(p_1) d\mu(p_2) a_{p_1}^{\text{in}\dagger} a_{p_2}^{\text{in}\dagger} |0\rangle e_{p_1} \otimes_{S_\theta} e_{p_2} \quad (78)$$

and

$$\int d\mu(p_1) d\mu(p_2) a_{p_1}^{\text{in}\dagger} a_{p_2}^{\text{in}\dagger} |0\rangle e^{ip_1 \wedge p_2} e_{p_2} \otimes_{S_\theta} e_{p_1} \quad (79)$$

requires that

$$a_{p_2}^{\text{in}\dagger} a_{p_1}^{\text{in}\dagger} |0\rangle = e^{-ip_1 \wedge p_2} a_{p_1}^{\text{in}\dagger} a_{p_2}^{\text{in}\dagger} |0\rangle \quad . \quad (80)$$

A solution is (73). It extends correctly to similar identities among N -fold tensor products. It thus seems

unique. We can then take (73) to be the relation between untwisted and twisted fields.

Suppose now that we transform $a_{p_1}^{\text{in}\dagger}$ and $a_{p_2}^{\text{in}\dagger}$ with $U_0(\Lambda)$:

$$U_0(\Lambda)a_{p_1}^{\text{in}\dagger}a_{p_2}^{\text{in}\dagger}U_0(\Lambda)^{-1}|0\rangle = a_{\Lambda p_1}^{\text{in}\dagger}a_{\Lambda p_2}^{\text{in}\dagger}|0\rangle e^{-\frac{i}{2}(\Lambda p_1)\wedge(\Lambda p_2)}e^{\frac{i}{2}p_1\wedge p_2} \quad . \quad (81)$$

This is as required by (72). This relation extends to N -particle states. Also since the Moyal \star -product is translationally invariant, $U_\theta(a) = U_0(a)$ for translation $a \equiv (a, \mathbf{1}) \in \mathcal{P}$. Hence we conclude that $U_0(g) \equiv U(g)$ implements the Poincaré group on the Hilbert space of states for the Moyal plane [27].

C. Self-Reproduction

If $\alpha, \beta \in C^\infty(\mathbb{R}^d)$ are scalar functions, then so is $\alpha\beta$ on the commutative spacetime $\mathcal{A}_0(\mathbb{R}^d)$. *Self-reproduction* extends this fact to $\mathcal{A}_\theta(\mathbb{R}^d)$.

The \star -product on the Moyal plane is given by

$$\alpha \star \beta = m_0\left(\mathcal{F}_\theta \alpha \otimes \beta\right), \quad (82)$$

with $m_0(\alpha \otimes \beta)$ is just the pointwise multiplication map $\alpha\beta$ of α and β . On plane waves

$$e_{p_1} \star e_{p_2} = e^{-\frac{i}{2}p_1\wedge p_2}e_{p_1+p_2} \quad (83)$$

Now consider

$$\varphi_\theta^{\text{in}} = \int d\mu(p) \left[a_p^{\text{in}\dagger} e_p + h.c. \right] = \varphi_0^{\text{in}} e^{-\frac{1}{2}\overleftarrow{\partial}\wedge P} \quad (84)$$

where we used (73).

If $\varphi_\theta^{\text{in}} \star \varphi_\theta^{\text{in}}$ is the \star -product of these fields (the \star being used in taking products of plane waves), and $\varphi_\theta^{\text{in}}\varphi_\theta^{\text{in}}$ denotes their point-wise product, then it follows that

$$\varphi_\theta^{\text{in}} \star \varphi_\theta^{\text{in}} = \left(\varphi_0^{\text{in}} \varphi_0^{\text{in}} \right) e^{-\frac{1}{2}\overleftarrow{\partial}\wedge P} \quad (85)$$

From this, we can easily prove that $\varphi_\theta^{\text{in}} \star \varphi_\theta^{\text{in}}$ transforms with the twisted coproduct under $U(g)$ just as $\varphi_\theta^{\text{in}}$ does. This extends to N -fold \star -products of $\varphi_\theta^{\text{in}}$.

The products of quantum fields of course have only a formal significance. Instead we can twist the functions

e_p to e_p^θ and implement (84):

$$\varphi_\theta^{\text{in}} = \int d\mu(p) \left[c_p^{\text{in}\dagger} e_p^\theta + h.c. \right], \quad e_p^\theta = e_p^0 e^{\frac{i}{2} p \wedge \mathcal{P}}, \quad \mathcal{P}_\mu = -i \partial_\mu \quad (86)$$

where $e_p^0 = e_p$. Then using (82), we get

$$e_{p_1}^\theta \star e_{p_2}^\theta = (e_{p_1}^0 e_{p_2}^0) e^{\frac{i}{2} (p_1 + p_2) \wedge \mathcal{P}} \quad (87)$$

or for generic twisted functions α^θ and β^θ ,

$$\alpha^\theta = \alpha e^{-\frac{1}{2} \overleftarrow{\partial} \wedge \mathcal{P}}, \quad \beta^\theta = \beta e^{-\frac{1}{2} \overleftarrow{\partial} \wedge \mathcal{P}}, \quad \alpha^\theta \star \beta^\theta = (\alpha \beta) e^{-\frac{1}{2} \overleftarrow{\partial} \wedge \mathcal{P}} \quad (88)$$

showing that this twist preserves the transformation properties of the Moyal algebra.

IX. COVARIANT QUANTUM FIELDS: GEONS ON COMMUTATIVE SPACETIMES

We assume that a covariant quantum field φ_0 can be associated with a geon when the underlying spacetime is commutative. Diffeomorphism invariance implies that $D_0^{(1)\infty}$ acts trivially on φ_0 . So the group $D^{(1)}/D_0^{(1)\infty}$ acts nontrivially on φ_0 by the pull-back of the action of $D^{(1)}$ on spacetime: if $g \in D/D_0^{\infty 2}$ and $\hat{g} = gg_0^\infty$, $g_0^\infty \in D_0^\infty$ is any member of the equivalence class gD_0^∞ , then

$$g : \varphi_0 \rightarrow g\varphi_0, \quad (g\varphi_0)(p) = \varphi_0(\hat{g}^{-1}p) \quad (89)$$

This action does not depend on the choice of g_0^∞ since $g_0^\infty \varphi_0 = \varphi_0$ for all $g_0^\infty \in D_0^\infty$, and hence is consistent.

Equation (89) has been written for scalar geon fields for simplicity. It is easily generalised to spinorial and tensorial fields.

Also for simplicity, we will henceforth write

$$(g\varphi_0)(p) = \varphi_0(g^{-1}p) \quad (90)$$

even though on the r.h.s., we should write $\hat{g}^{-1}p$.

² From now on we will only refer to the single geon diffeo group. Therefore we will use D , D^∞ and D_0^∞ instead of $D^{(1)}$, $D^{(1)\infty}$ and $D_0^{(1)\infty}$ to simplify the notation.

Covariance implies that there exists a representation U of D/D_0^∞ so that

$$U(g)\varphi_0(g^{-1}p)U(g)^{-1} = \varphi_0(p) \quad (91)$$

or

$$U(g)\varphi_0(p)U(g)^{-1} = \varphi_0(gp) \quad . \quad (92)$$

The twist we now consider is based on abelian discrete compact groups A : as we saw, its dependence on representations of \mathbb{Z} is trivial. Let $f_{\vec{m}}^{(\pm)}$ furnish the orthonormal basis on the geon spacetime which carry the UIRR $\vec{m} = (m_1, m_2, \dots, m_k)$ of $A = \times_{i=1}^k \mathbb{Z}_{n_i}$ and which have positive and negative frequencies $\pm|E_{\vec{m}}|^3$:

$$f_{\vec{m}}^{(\pm)}(h^{-1}p) = f_{\vec{m}}^{(\pm)}(p)\chi_{\vec{m}}(h), \quad h \in A \quad , \quad (93)$$

$$i\partial_0 f_{\vec{m}}^{(\pm)} = \pm|E_{\vec{m}}|f_{\vec{m}}^{(\pm)} \quad (94)$$

Here $\chi_{\vec{m}}$ is the character function of A . Since $\bar{\chi}_{\vec{m}} = \chi_{-\vec{m}}$, we can assume that

$$\bar{f}_{\vec{m}}^{(\pm)} = f_{-\vec{m}}^{(\mp)} \quad (95)$$

If $g \in D/D_0^\infty$, we can then write

$$f_{\vec{m}}^{(\pm)}(g^{-1}p) = \sum_{\vec{m}'} f_{\vec{m}'}^{(\pm)}(p) \mathcal{D}_{\vec{m}'\vec{m}}(g) \quad (96)$$

where \mathcal{D} is a unitary representation of D/D_0^∞ .

The untwisted quantum field φ_0 , assumed real for simplicity, and also assumed to be in, out or free field, can be written as

$$\varphi_0 = \sum_{\vec{m}} \left[c_{\vec{m}} f_{\vec{m}}^{(+)} + c_{\vec{m}}^\dagger f_{-\vec{m}}^{(-)} \right] \quad (97)$$

Here $c_{\vec{m}}, c_{\vec{m}}^\dagger$ are annihilation and creation operators:

$$[c_{\vec{m}}, c_{\vec{n}}^\dagger] = \delta_{\vec{m}, \vec{n}} \quad , \quad (98)$$

$$[c_{\vec{m}}, c_{\vec{n}}] = [c_{\vec{m}}^\dagger, c_{\vec{n}}^\dagger] = 0 \quad . \quad (99)$$

³ We assume their existence as is normally the case.

Covariance is the requirement that there is a unitary representation of D/D_0^∞ on the Hilbert space of vector states such that

$$U(g)\varphi_0(g^{-1}p)U(g)^{-1} = \varphi_0(p) \quad . \quad (100)$$

Hence since $\bar{\mathcal{D}}_{\vec{m}'\vec{m}}(g)\mathcal{D}_{\vec{n}'\vec{m}}(g) = \delta_{\vec{m}',\vec{n}'}$ (with sum over \vec{m} being implicit),

$$U(g)c_{\vec{m}}U(g)^{-1} = c_{\vec{m}'}\bar{\mathcal{D}}_{\vec{m}'\vec{m}}(g) \quad , \quad (101)$$

$$U(g)c_{\vec{m}}^\dagger U(g)^{-1} = c_{\vec{m}'}^\dagger \mathcal{D}_{\vec{m}'\vec{m}}(g) \quad . \quad (102)$$

For untwisted fields, the symmetrisation postulates on $f_{\vec{m}}^{(\pm)}$ are based on Bose statistics for tensorial fields. They are incorporated in (98,99) and are compatible with covariance.

A. Covariance for Abelian Twists

The twisted quantum field φ_θ associated with φ_0 is written as

$$\varphi_\theta = \sum_{\vec{m}} \left[a_{\vec{m}} f_{\vec{m}}^{(+)} + a_{\vec{m}}^\dagger f_{-\vec{m}}^{(-)} \right] \quad (103)$$

We will as before deduce the relation of $a_{\vec{m}}, a_{\vec{m}}^\dagger$ to $c_{\vec{m}}, c_{\vec{m}}^\dagger$ using covariance.

First consider

$$\varphi_\theta^{(-)}|0\rangle = \sum_{\vec{m}} a_{\vec{m}}^\dagger|0\rangle f_{-\vec{m}}^{(-)} = \sum_{\vec{m}} a_{\vec{m}}^\dagger|0\rangle \bar{f}_{\vec{m}}^{(+)} \quad . \quad (104)$$

Covariance implies the requirement

$$\sum_{\vec{m}} U(g)a_{\vec{m}}^\dagger U(g)^{-1}|0\rangle \bar{f}_{\vec{m}'}^{(+)} \bar{\mathcal{D}}_{\vec{m}'\vec{m}}(g) = \varphi_\theta^{(-)}|0\rangle \quad (105)$$

where $U(g)$ represents g on the vector states and we use $U(g)|0\rangle = |0\rangle$. Hence

$$U(g)a_{\vec{m}}^\dagger U(g)^{-1}|0\rangle = \sum_{\vec{m}''} a_{\vec{m}''}^\dagger|0\rangle \mathcal{D}_{\vec{m}''\vec{m}}(g) \quad (106)$$

Next consider the two-particle case:

$$\varphi_\theta^{(-)} \otimes \varphi_\theta^{(-)} |0\rangle = \sum_{\vec{m}, \vec{n}} a_{\vec{m}}^\dagger a_{\vec{n}}^\dagger |0\rangle \bar{f}_{\vec{m}}^{(+)} \otimes \bar{f}_{\vec{n}}^{(+)} \quad (107)$$

The action of $g \in D/D_0^\infty$ on $\bar{f}_{\vec{m}}^{(+)} \otimes \bar{f}_{\vec{n}}^{(+)}$ is via the twisted coproduct:

$$\begin{aligned} g \triangleright \bar{f}_{\vec{m}}^{(+)} \otimes \bar{f}_{\vec{n}}^{(+)} &= F_\theta^{-1}(g \otimes g) F_\theta \bar{f}_{\vec{m}}^{(+)} \otimes \bar{f}_{\vec{n}}^{(+)} \\ &= F_\theta^{-1}(g \otimes g) \bar{f}_{\vec{m}}^{(+)} \otimes \bar{f}_{\vec{n}}^{(+)} e^{-\frac{i}{2} m_i \theta_{ij} n_j} \\ &= F_\theta^{-1} \sum_{\vec{m}', \vec{n}'} \bar{f}_{\vec{m}'}^{(+)} \otimes \bar{f}_{\vec{n}'}^{(+)} \bar{\mathcal{D}}_{\vec{m}' \vec{m}}(g) \bar{\mathcal{D}}_{\vec{n}' \vec{n}}(g) e^{-\frac{i}{2} m_i \theta_{ij} n_j} \\ &= \sum_{\vec{m}', \vec{n}'} \bar{f}_{\vec{m}'}^{(+)} \otimes \bar{f}_{\vec{n}'}^{(+)} e^{\frac{i}{2} m'_i \theta_{ij} n'_j} \bar{\mathcal{D}}_{\vec{m}' \vec{m}}(g) \bar{\mathcal{D}}_{\vec{n}' \vec{n}}(g) e^{-\frac{i}{2} m_i \theta_{ij} n_j} \quad . \end{aligned} \quad (108)$$

The covariance requirement

$$\sum_{\vec{m}, \vec{n}} U(g) a_{\vec{m}}^\dagger a_{\vec{n}}^\dagger |0\rangle \left(g \triangleright \bar{f}_{\vec{m}}^{(+)} \otimes \bar{f}_{\vec{n}}^{(+)} \right) = \varphi_\theta^{(-)} \otimes \varphi_\theta^{(-)} |0\rangle \quad (109)$$

can thus be fulfilled by setting

$$a_{\vec{m}}^\dagger = \sum_{\vec{m}'} c_{\vec{m}}^\dagger e^{\frac{i}{2} m_i \theta_{ij} m'_j} \mathbb{P}_{\vec{m}'} \quad (110)$$

and identifying $U(g)$ as the untwisted operator with the action (101,102) on $c_{\vec{m}}, c_{\vec{m}}^\dagger$.

The adjoint of (110) gives

$$a_{\vec{m}} = \sum_{\vec{m}'} \left(e^{-\frac{i}{2} m_i \theta_{ij} m'_j} \mathbb{P}_{\vec{m}'} \right) c_{\vec{m}} \equiv V_{-\vec{m}} c_{\vec{m}} \quad (111)$$

Now $V_{-\vec{m}}$ is unitary with inverse

$$V_{-\vec{m}}^{-1} = V_{\vec{m}} = \sum_{\vec{m}'} e^{\frac{i}{2} m_i \theta_{ij} m'_j} \mathbb{P}_{\vec{m}'} \quad (112)$$

It is the unitary operator on the quantum Hilbert space representing the element

$$\times_j e^{\frac{i}{2} m_i \theta_{ij}} \in A \quad (113)$$

(The quantisation condition on θ_{ij} is also manifest from here.) Hence

$$V_{\vec{m}} a_{\vec{m}} V_{\vec{m}}^{-1} = e^{-\frac{i}{2} m_i \theta_{ij} m_j} a_{\vec{m}} = a_{\vec{m}} \quad (114)$$

and

$$a_{\vec{m}} = \sum_{\vec{m}'} c_{\vec{m}} e^{-\frac{i}{2} m_i \theta_{ij} m_j'} \mathbb{P}_{m_j'} \quad (115)$$

so that we can freely twist on left or right.

The twisted symmetrisation properties (statistics) of the multigeon states

$$a_{\vec{m}_1}^\dagger a_{\vec{m}_2}^\dagger \dots a_{\vec{m}_N}^\dagger |0\rangle \quad (116)$$

follows from (110).

Self-reproduction under the \star -product can also be easily verified:

$$(a_{\vec{m}} f_{\vec{m}}^{(+)}) \star (a_{\vec{n}} f_{\vec{n}}^{(+)}) = \left(c_{\vec{m}} c_{\vec{n}} \right) e^{\frac{i}{2} (m_i + n_i) \theta_{ij} m_j'} f_{\vec{m}+\vec{n}}^{(+)} \mathfrak{P}_{m_j'} \quad (117)$$

There are similar equation involving creation operators. Here again $\mathfrak{P}_{\vec{m}} = \prod_j \mathfrak{P}_{m_j}$ be the projection operator which acting on functions, projects out the IRR \vec{m} of A . With this notation, we can incorporate the dressing transformation directly in φ_θ :

$$\varphi_\theta = \sum_{\vec{m}, \vec{m}'} \left(\mathfrak{P}_{\vec{m}} \varphi \right) e^{-\frac{i}{2} m_i \theta_{ij} m_j'} \mathbb{P}_{\vec{m}'} \quad . \quad (118)$$

This equation is the analogue of the dressing transformation for the Moyal field.

X. HOW MAY WE GENERALISE?

Physical considerations outlined below suggest that the twist discussed above (and its generalisations such as that in the Moyal case) is unique upto unitary equivalence if we require the spacetime algebra to be associative. We do have nonassociative examples [28], they are associated with quasi-Hopf algebras as symmetries. We will now briefly consider them as well.

A. Abelian Twists \Rightarrow Associative Spacetimes

For the abelian algebra, we retain $A = \times_{i=1}^k \mathbb{Z}_{n_i}$. If $f_{\vec{m}}^{(\eta)}$, ($\eta = \pm$), denote the same functions as before, then for the \star -product, we assume the general form

$$f_{\vec{m}}^{(\eta)} \star f_{\vec{m}'}^{(\varrho)} = \sigma(\vec{m}, \vec{m}') f_{\vec{m}}^{(\eta)} f_{\vec{m}'}^{(\varrho)}, \quad \eta, \varrho = \pm, \quad \sigma(\vec{m}, \vec{m}') \in \mathbb{C} \quad , \quad (119)$$

where on the right, $f_{\vec{m}}^{(\eta)} f_{\vec{m}'}^{(\varrho)}$ denotes point-wise product.

Now $f_{\vec{m}}^{(\eta)} f_{\vec{m}'}^{(\varrho)}$ transforms by the representation $\vec{m} + \vec{m}'$ (modulo n_i in each entry). Taking this into account we require associativity:

$$f_{\vec{m}}^{(\eta)} \star \left(f_{\vec{m}'}^{(\varrho)} \star f_{\vec{m}''}^{(\zeta)} \right) = \left(f_{\vec{m}}^{(\eta)} \star f_{\vec{m}'}^{(\varrho)} \right) \star f_{\vec{m}''}^{(\zeta)} \quad . \quad (120)$$

The l.h.s. and r.h.s. of this equation are

$$\text{l.h.s.} = \sigma(\vec{m}, \vec{m}' + \vec{m}'') \sigma(\vec{m}', \vec{m}'') f_{\vec{m}}^{(\eta)} f_{\vec{m}'}^{(\varrho)} f_{\vec{m}''}^{(\zeta)} \quad (121)$$

$$\text{r.h.s.} = \sigma(\vec{m}, \vec{m}') \sigma(\vec{m} + \vec{m}', \vec{m}'') f_{\vec{m}}^{(\eta)} f_{\vec{m}'}^{(\varrho)} f_{\vec{m}''}^{(\zeta)} \quad (122)$$

Therefore

$$\sigma(\vec{m}, \vec{m}' + \vec{m}'') \sigma(\vec{m}', \vec{m}'') = \sigma(\vec{m}, \vec{m}') \sigma(\vec{m} + \vec{m}', \vec{m}'') \quad . \quad (123)$$

It has the solution

$$\sigma(\vec{m}, \vec{m}') = e^{-\frac{i}{2} m_i \hat{\theta}_{ij} m'_j} \quad (124)$$

where $\hat{\theta}_{ij}$ is quantised as before:

$$\hat{\theta}_{ij} = \frac{4\pi}{n_{ij}}, \quad \frac{n_i}{n_{ij}}, \frac{n_j}{n_{ij}} \in \mathbb{Z} \quad . \quad (125)$$

Note that the quantisation requirement forces $\hat{\theta}_{ij}$ to be real, but not necessarily antisymmetric. Hence we can in general write

$$\hat{\theta}_{ij} = \theta_{ij} + s_{ij}, \quad \theta_{ij} = -\theta_{ji} = \frac{4\pi}{n_{ij}}, \quad s_{ij} = s_{ji} = \frac{4\pi}{m_{ij}} \quad (126)$$

where both n_{ij} and m_{ij} divide n_i and n_j , that is fulfill the analogue of (41).

Thus associativity and quantisation conditions reduce σ to the form

$$\sigma(\vec{m}, \vec{m}') = e^{-\frac{i}{2}m_i\theta_{ij}m'_j} e^{\frac{i}{2}m_is_{ij}m'_j} \quad (127)$$

with the constraints on θ_{ij} and s_{ij} stated above.

The corresponding Drinfel'd twist is

$$F_\sigma = \sum_{\vec{m}, \vec{m}'} \sigma(\vec{m}, \vec{m}') \mathbb{P}_{\vec{m}} \otimes \mathbb{P}_{\vec{m}'} \quad (128)$$

Note that

$$|\sigma(\vec{m}, \vec{m}')| = 1, \quad F_\sigma^{-1} = F_{\bar{\sigma}} \quad . \quad (129)$$

If ϵ is the counit, then there is the normalisation condition [4]

$$(\epsilon \otimes \mathbb{1})F_\sigma = (\mathbb{1} \otimes \epsilon)F_\sigma = \mathbb{1} \quad . \quad (130)$$

Here ϵ is the map to the “trivial” representation, so \vec{m} and \vec{m}' become $\vec{0} \pmod{\vec{n} = (n_1, \dots, n_k)}$ under ϵ and $\epsilon(\mathbb{P}_{\vec{m}}) = \delta_{\vec{m}, \vec{0}}$, $\epsilon(\mathbb{P}_{\vec{m}'}) = \delta_{\vec{m}', \vec{0}}$. Since $\sum_{\vec{m}} \mathbb{P}_{\vec{m}} = \sum_{\vec{m}'} \mathbb{P}_{\vec{m}'} = \mathbb{1}$, the above requirement is fulfilled by (128).

Next we show that the symmetric factor with s_{ij} can be eliminated by requiring that the twist preserves the adjoint operation.

For the twist F_σ above, the dressed annihilation and creation operators are

$$a_{\vec{m}} = \sum_{\vec{m}'} c_{\vec{m}'} e^{-\frac{i}{2}m_i(\theta_{ij}+s_{ij})m'_j} \mathbb{P}_{\vec{m}'} \quad (131)$$

$$a_{\vec{m}}^* = \sum_{\vec{m}'} c_{\vec{m}}^\dagger e^{\frac{i}{2}m_i(\theta_{ij}+s_{ij})m'_j} \mathbb{P}_{\vec{m}'} \quad (132)$$

where $*$ denotes that it is not necessarily the adjoint † of $a_{\vec{m}}$, and we have used the fact that $a_{\vec{m}}^*$ transforms by the representation $-\vec{m}$.

Now

$$a_{\vec{m}}^\dagger = \left(\sum_{\vec{m}'} e^{\frac{i}{2}m_i(\theta_{ij}+s_{ij})m'_j} \mathbb{P}_{\vec{m}'} \right) c_{\vec{m}}^\dagger \quad (133)$$

The prefactor is the unitary operator $U_{\vec{m}}$ representing the element

$$\times_j e^{\frac{i}{2}m_i(\theta_{ij}+s_{ij})} \quad (134)$$

in A . Hence

$$U_{\vec{m}}c_{\vec{m}}^\dagger U_{\vec{m}}^{-1} = e^{\frac{i}{2}m_i(\theta_{ij}+s_{ij})m_j}c_{\vec{m}}^\dagger = e^{\frac{i}{2}m_i s_{ij}m_j}c_{\vec{m}}^\dagger \quad (135)$$

since $\theta_{ij} = -\theta_{ji}$. Thus

$$a_{\vec{m}}^\dagger = e^{\frac{i}{2}m_i s_{ij}m_j}a_{\vec{m}}^* \quad . \quad (136)$$

The requirement

$$a_{\vec{m}}^* = a_{\vec{m}}^\dagger \quad (137)$$

imposes the constraint

$$e^{\frac{i}{2}m_i s_{ij}m_j} = 1 \quad (138)$$

From this we can infer that $s_{ij} = 0 \bmod 4\pi/m_{ij}$ where $n_i/m_{ij}, n_j/m_{ij} \in \mathbb{Z}$. For example the successive choices $\vec{m} = (1, \mathbf{0}), (0, 1, \vec{0}), (1, 1, \vec{0})$, shows that $s_{ij} = 0$ if $i, j \leq 2$. Thus (138) reduces σ to

$$\sigma(\vec{m}, \vec{m}') = e^{-\frac{i}{2}m_i \theta_{ij} m'_j} \quad (139)$$

It thus appears that our previous considerations are general for associative spacetime algebras.

B. Nonabelian Generalisations of Drinfel'd Twists

We now discuss nonabelian generalisations of the above considerations. They generally lead to quasi-Hopf algebras based on D^∞/D_0^∞ as the symmetry algebras and non-associative spacetimes.

Here is an approach to such a generalisation. Let us consider the following nested groups:

$$D^\infty/D_0^\infty \equiv G_0 \supset G_1 \supset \dots \supset G_N = A \quad (140)$$

Here $A = \times_{i=1}^k \mathbb{Z}_{n_i}$ is the maximal abelian subgroup of G_0 (quotiented by factors of \mathbb{Z}) while the rest, G_k for $k < N$, can be nonabelian. The chain is supposed to be such that there exists an orthonormal basis $\{b(\vec{\varrho})\}$

$(\vec{\varrho} = (\varrho_0, \varrho_1, \dots, \varrho_N))$ for the vector space $V^{(\varrho_0)}$ for the IRR ϱ_0 of G_0 where $b^{(\vec{\varrho})}$ is a vector in the representation space for the IRR ϱ_j of G_j . In this notation, $\varrho_N =$ our previous \vec{m} . Thus the chain (140) leads to a complete system of labels for the basis vectors.

Let $\mathbb{P}_{\vec{\varrho}}$ be the projector to the space $\mathbb{C}b^{(\vec{\varrho})}$:

$$\mathbb{P}_{\vec{\varrho}} b^{(\vec{\varrho})} = b^{(\vec{\varrho})} \quad . \quad (141)$$

Then

$$\mathbb{P}_{\vec{\varrho}} \mathbb{P}_{\vec{\varrho}'} = \delta_{\vec{\varrho}, \vec{\varrho}'} \mathbb{P}_{\vec{\varrho}}, \quad \sum_{\vec{\varrho}} \mathbb{P}_{\vec{\varrho}} = \mathbf{1} \quad . \quad (142)$$

Let $\vec{\varrho}_\epsilon$ label the IRR associated with the counit ϵ . Then

$$\epsilon(\mathbb{P}_{\vec{\varrho}}) = \delta_{\vec{\varrho}, \vec{\varrho}_\epsilon} \quad (143)$$

Now consider

$$F_\sigma = \sum_{\vec{\varrho}, \vec{\varrho}'} \sigma(\vec{\varrho}, \vec{\varrho}') \mathbb{P}_{\vec{\varrho}} \otimes \mathbb{P}_{\vec{\varrho}'}, \quad \sigma(\vec{\varrho}, \vec{\varrho}') \in \mathbb{C} \quad . \quad (144)$$

We plan to use F_σ as the Drinfel'd twist. Its realization used to deform the \star -product of functions will be indicated as usual as \mathcal{F}_σ . It involves the realization of $\mathbb{P}_{\vec{\varrho}}$'s on functions which again we will call $\mathfrak{P}_{\vec{\varrho}}$'s. The Drinfel'd twist of the coproduct as in (20), requires F_σ to be invertible so that

$$\sigma(\vec{\varrho}, \vec{\varrho}') \neq 0 \quad \text{for any } \vec{\varrho}, \vec{\varrho}' \quad . \quad (145)$$

For then,

$$F_\sigma^{-1} = \sum_{\vec{\varrho}, \vec{\varrho}'} \frac{1}{\sigma(\vec{\varrho}, \vec{\varrho}')} \mathbb{P}_{\vec{\varrho}} \otimes \mathbb{P}_{\vec{\varrho}'} \quad . \quad (146)$$

The next requirement on F_σ is the normalisation condition

$$(\epsilon \otimes \mathbf{1})F_\sigma = (\mathbf{1} \otimes \epsilon)F_\sigma = \mathbf{1} \quad . \quad (147)$$

In view of (144), this requires that

$$\sigma(\vec{\varrho}_\epsilon, \vec{\varrho}) = \sigma(\vec{\varrho}, \vec{\varrho}_\epsilon) = 1 \quad . \quad (148)$$

According to Majid [4], there is no further requirement on F_σ if quasi-Hopf algebras are acceptable. The

spacetime algebra with its star product governed by F_σ as in previous sections is then its module algebra which is generally nonassociative (with an associator) [29–31]. It is associative only if its symmetry algebra is Hopf.

The spacetime orthonormal basis is now denoted by $b_{\vec{\varrho}}^{(\pm)}$ instead of by $f_{\vec{m}}^{(\pm)}$ while the twisted quantum field is written as

$$\varphi_\theta = \sum_{\vec{\varrho}, \vec{\varrho}'} \left[a_{\vec{\varrho}} b_{\vec{\varrho}}^{(+)} + a_{\vec{\varrho}}^* b_{\vec{\varrho}}^{(-)} \right] \quad (149)$$

$$a_{\vec{\varrho}} = \sum_{\vec{\varrho}'} c_{\vec{\varrho}} \sigma(\vec{\varrho}, \vec{\varrho}') \mathbb{P}_{\vec{\varrho}'} \quad (150)$$

$$a_{\vec{\varrho}}^* = \sum_{\vec{\varrho}'} c_{\vec{\varrho}}^\dagger \bar{\sigma}(\vec{\varrho}, \vec{\varrho}') \mathbb{P}_{\vec{\varrho}'} \quad (151)$$

where $c_{\vec{\varrho}}, c_{\vec{\varrho}}^\dagger$ are the untwisted annihilation and creation operators.

Unitarity requires that

$$a_{\vec{\varrho}}^* = a_{\vec{\varrho}}^\dagger = \sum_{\vec{\varrho}'} \bar{\sigma}(\vec{\varrho}, \vec{\varrho}') \mathbb{P}_{\vec{\varrho}'} c_{\vec{\varrho}}^\dagger = \sum_{\vec{\varrho}'} c_{\vec{\varrho}}^\dagger \bar{\sigma}(\vec{\varrho}, \vec{\varrho}') \mathbb{P}_{\vec{\varrho}'} \quad (152)$$

Hence we have also

$$\sigma(\vec{\varrho}, \vec{\varrho}) = 1 \quad (153)$$

Thus it appears that we have an approach to a quantum field theory if the normalisation condition (148) and the unitary condition (153) are fulfilled.

If $\mathfrak{P}_{\vec{\varrho}}$ is the projector on the space of functions to the IRR $\vec{\varrho}$, the twisted field can be written without a mode expansion:

$$\varphi_\theta = \sum_{\vec{\varrho}, \vec{\varrho}'} \sigma(\vec{\varrho}, \vec{\varrho}') \left(\mathfrak{P}_{\vec{\varrho}} \varphi_0 \right) \mathbb{P}_{\vec{\varrho}'} \quad (154)$$

It is then easily verified that the dressed field (154) coincides with (149) and it has the self-reproducing property:

$$\varphi_\theta \star \varphi_\theta = \sum_{\vec{\varrho}, \vec{\varrho}'} \sigma(\vec{\varrho}, \vec{\varrho}') \left(\mathfrak{P}_{\vec{\varrho}} \varphi_0^2 \right) \mathbb{P}_{\vec{\varrho}'} \quad (155)$$

But there is in general no associativity:

$$(\varphi_\theta \star \varphi_\theta) \star \varphi_\theta \neq \varphi_\theta \star (\varphi_\theta \star \varphi_\theta) \quad (156)$$

Such quantum fields merit study. They seem to lead to Pauli principle violations with testable experimental

consequences. We will elaborate on this remark elsewhere.

XI. ON RINGS AND THEIR STATISTICS (MOTION) GROUPS

A theoretical approach to the investigation of statistics of a system of identical constituents is based on the properties of the fundamental group of its configuration space. For N spinless identical particles in a Euclidean space of three or more dimensions, for example, this group is known to be the permutation group S_N [18]. There is furthermore an orderly method for the construction of a distinct quantum theory for each of its unitary irreducible representations (UIRs). As these theories describe bosons, fermions and paraparticles according to the choice of the representation, the study of the fundamental group leads to a comprehensive account of the possible statistics of structureless particles in three or more dimensions.

It has been appreciated for some time that the statistical possibilities of a particle species confined to the plane \mathbb{R}^2 can be quite different from those in three or more dimensions. This is because the fundamental group for N identical spinless particles in a plane is not S_N . It is instead an infinite group B_N , known as the braid group. Since S_N is a factor group of B_N , and hence representations of S_N are also those of B_N , it is of course possible to associate Bose, Fermi or parastatistics with a particle species in a plane. But since B_N has many more UIRs which are not UIRs of S_N , there are also several possibilities for exotic planar statistics. One such possibility of particular interest, for instance, is that of fractional statistics, which is of importance in the context of fractional quantum Hall effect.

As we discussed earlier, it was pointed out some time ago that configuration space with unusual fundamental groups, and hence exotic statistical possibilities, occur not merely for point particles on a plane, but also for topological geons. It was also emphasized elsewhere [21] that there are many remarkable properties associated with the quantum version of geons, such as the failure of the spin-statistics connection and the occurrence of states in three spatial dimensions which are not bosons, fermions or paraparticles.

In [15, 32] the investigation of exotic statistics was continued by examining another system of extended objects, namely a system of identical closed strings assumed to be unknots and imbedded in three spatial dimensions. Using known mathematical results on motion groups [33], it was shown that the fundamental group of the configuration space of two or more such strings is not the permutation group either. It is instead an infinite non-Abelian group which bears a certain resemblance to the gravitational fundamental groups

mentioned a moment ago. It was further shown that quantum strings as well may not be characterized by permutation group representations. Thus they may not obey Bose, Fermi or parastatistics. They may also fail to obey the familiar spin-statistics connection.

Thus identical geons and identical knots share certain topological properties. For this reason, in this section we briefly examine the statistics of identical unknots. We here consider only the configuration spaces of one and two unknots and their fundamental groups.

We denote the configuration space of N unknots in \mathbb{R}^3 as $\mathcal{Q}^{(N)}$ and consider $N = 1$ and 2 . These unknots can be unoriented or oriented. These cases will be discussed separately.

A. *The case of one unoriented unknot*

An unoriented unknot is a map of a circle S^1 into \mathbb{R}^3 where the image is the unoriented unknot. That means the following:

- a) It can be deformed to the standard map where the image is say the circle $\{(x, y, 0) : \sum x^2 + y^2 = 1\}$ in the 1-2 plane. (Here we chose the flat metric δ_{ij}).
- b) Two maps which differ by an orientation reversal of S^1 are identified.

Intuitively, an unknot is a closed loop deformable to the above standard loop.

The configuration space $\mathcal{Q}^{(1)}$ of the unknot consists of all such maps.

We now consider the fundamental group $\pi_1(\mathcal{Q}^{(1)})$.

The construction of $\pi_1(\mathcal{Q}^{(1)})$ involves the choice of a fixed (“base”) point \bar{q} in $\mathcal{Q}^{(1)}$. As $\mathcal{Q}^{(1)}$ is the space of maps from S^1 to \mathbb{R}^3 , \bar{q} in this case is one particular choice of such maps. If $\mathcal{Q}^{(1)}$ is connected, as is the case for us, it can be any point \bar{q} of $\mathcal{Q}^{(1)}$. The resultant group $\pi_1(\mathcal{Q}^{(1)}, \bar{q})$, where we have put in the base point \bar{q} in the notation for the fundamental group, does not depend on \bar{q} . So we can talk of $\pi_1(\mathcal{Q}^{(1)})$ and omit \bar{q} .

But there is no canonical isomorphism between $\pi_1(\mathcal{Q}^{(1)}; \bar{q})$ and $\pi_1(\mathcal{Q}^{(1)}; \bar{q}')$ with $\bar{q} \neq \bar{q}'$. Any isomorphism depends on the choice of the path from \bar{q} to \bar{q}' [18].

For \bar{q} , we can for convenience choose the flat metric δ_{ij} in \mathbb{R}^3 as we did above, and choose \bar{q} to be a circle of unit radius centered at the origin in the $x - y$ plane as in Fig. 4 (a).

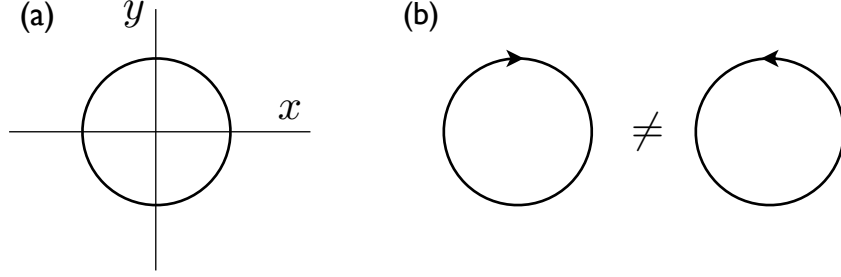


FIG. 4. (a) Circle of unit radius centered at the origin. (b) Two oriented inequivalent knots.

Consider rotating this figure by π around the x -axis. It maps \bar{q} to \bar{q} and creates a loop T in $\mathcal{Q}^{(1)}$ as the rotation evolves from 0 to π . The loop cannot be deformed to a point, the point loop based at \bar{q} . So $[T]$, the homotopy class of this loop is a non-trivial element of $\pi_1(\mathcal{Q}^{(1)})$.

Rotating \bar{q} around another axis \hat{n} ($\hat{n} \cdot \hat{n} = 1$) through the origin generates a loop which however is homotopic to T : just consider the sequence of loops got by rotating \hat{n} to the x -axis \hat{i} to this result.

By repeating T k -times, we get a $k\pi$ rotation loop call it T^k , of \bar{q} . If J_1 is the angular momentum of $SO(3)$, then

$$\{e^{i\theta J_1} : 0 \leq \theta \leq 2\pi\} \quad (157)$$

is a 2π -rotation loop in $SO(3)$ and T^2 is just $\{e^{i\theta J_1} \bar{q} : 0 \leq \theta \leq 2\pi\}$.

But this loop can be deformed to a point. For consider the sequence of loops

$$\{e^{i\theta \hat{n} \cdot \vec{J}} \bar{q} : 0 \leq \theta \leq 2\pi, \hat{n} \cdot \hat{n} = 1\} \quad (158)$$

as \hat{n} varies from $(1,0,0)$ to $(0,0,1)$. The starting loop is T , the final loop is a point. Thus $[T^2]=e$.

We thus see that

$$\pi_1(\mathcal{Q}^{(1)}) = \mathbb{Z}_2 = \langle [T], [T^2] = e \rangle \quad (159)$$

B. The case of the oriented unknot

In this case we drop the identification $b)$ above so that there is an arrow attached to the unknot like in Fig. 4 (b). Otherwise, its configuration space $\mathcal{Q}^{(1)}$ is defined as above.

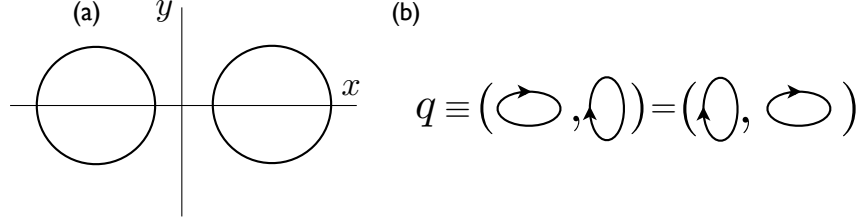


FIG. 5. (a) The configuration space of two identical unoriented unknots. (b) Pair of unordered identical knots.

As for $\pi_1(\mathcal{Q}^{(1)})$, the base point \bar{q} is as above, but there is now an arrow on the circle in Fig. 4 (b). Hence the curve

$$T = \langle e^{i\theta J_1} \bar{q} : 0 \leq \theta \leq \pi \rangle \quad (160)$$

does not close (is not a loop). We conclude that

$$\pi_1(\mathcal{Q}^{(1)}) = \{e\} \quad (161)$$

C. The case of two identical unoriented unknots

Its configuration space $\mathcal{Q}^{(2)}$ can be informally described (see Fig. 5) (a) as follows: A point $q \in \mathcal{Q}^{(2)}$ consists now of 2 unlinked unknots in \mathbb{R}^3 . The pair is unordered as the knots are “identical”, see Fig. 5 (b). This requirement is as for identical particles [18].

For \bar{q} , using our flat metric, we choose two circles of unit radius on the $x - y$ plane centered in ± 2 .

The discussion of identical unknots here is to be compared with the corresponding discussion of identical geons [21].

We can now recognize the following elements of $\pi_1(\mathcal{Q}^{(2)})$:

- *Exchange* [E]: The loop E defining “exchange” rotates \bar{q} from 0 to π around third axis:

$$E = \langle e^{i\theta J_3} \bar{q} : 0 \leq \theta \leq \pi \rangle \quad (162)$$

Its evolving pictures are depicted in Fig. 6.

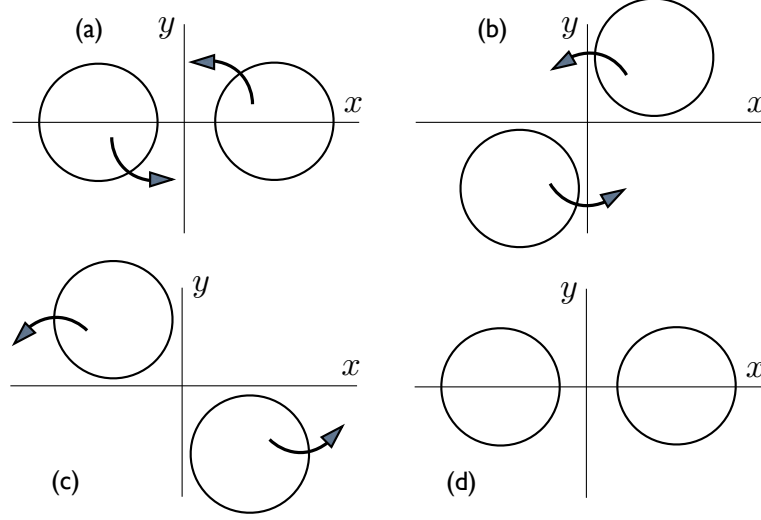


FIG. 6. The loop E defining “exchange”.

There are standard proofs that

$$E^2 \equiv \langle e^{i\theta J_3} \bar{q} : 0 \leq \theta \leq 2\pi \rangle \quad (163)$$

is deformable to e and that the loop with $\theta \rightarrow -\theta$ in (163) is homotopic to E .

The homotopy class $[E]$ of E in $\pi_1(\mathcal{Q}^{(2)})$ is the exchange. The corresponding group is S_2 .

- *The π -rotations* $[T^{(1)}], [T^{(2)}]$.

The loop $T^{(1)}$ rotates the ring 1 (on left) by π around x -axis, $T^{(2)}$ does so for the ring 2 on right. They are inherited from $\mathcal{Q}^{(1)}$ and generate the elements $[T^{(i)}]$ in $\pi_1(\mathcal{Q}^{(2)})$. Clearly

$$[E][T^{(1)}][E^{-1}] = [T^{(2)}], \quad [E][T^{(2)}][E^{-1}] = [T^{(1)}] \quad (164)$$

where the products in $\pi_1(\mathcal{Q}^{(2)})$ are as usual defined by concatenation of loops in $\mathcal{Q}^{(2)}$.

$[T^{(1)}]$ and $[T^{(2)}]$ commute.

- *The Slide:* Let us first consider the loop \mathcal{S}_{12} or the slide $[\mathcal{S}_{12}]$ of 2 through 1. Figure 7 above explains the loop:

The homotopic class $[\mathcal{S}_{12}]$ of \mathcal{S}_{12} is the slide of 2 through 1.

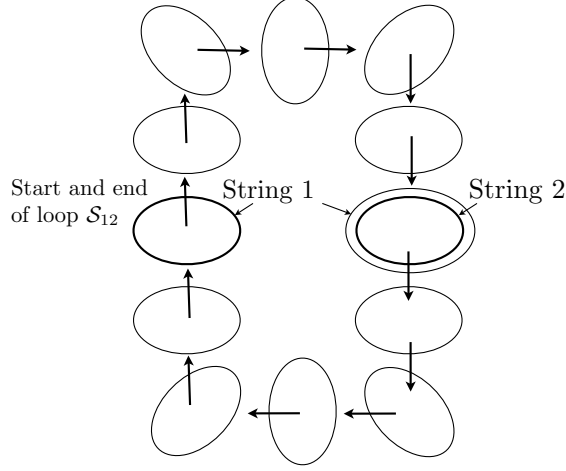


FIG. 7. The “slide” loop \mathcal{S}_{12} .

The slide $[\mathcal{S}_{21}]$ of 1 through 2 is similarly defined. We can show that

$$[\mathbf{E}][\mathcal{S}_{12}][\mathbf{E}]^{-1} = [\mathcal{S}_{21}] \quad , \quad (165)$$

$$[\mathbf{E}][\mathcal{S}_{21}][\mathbf{E}]^{-1} = [\mathcal{S}_{12}] \quad . \quad (166)$$

The full group $\pi_1(\mathcal{Q}^{(2)})$ is thus generated by $[\mathbf{T}^{(1)}]$, $[\mathbf{T}^{(2)}]$, $[\mathbf{E}]$, $[\mathcal{S}_{12}]$, $[\mathcal{S}_{21}]$ with the relations

$$[\mathbf{T}^{(1)}]^2 = [\mathbf{E}]^2 = [\mathbf{T}^{(2)}]^2 = e, [\mathbf{E}][\mathbf{T}^{(1)}][\mathbf{E}]^{-1} = [\mathbf{T}^{(2)}], [\mathbf{T}^{(1)}][\mathbf{T}^{(2)}] = [\mathbf{T}^{(2)}][\mathbf{T}^{(1)}], \quad (167)$$

and (165,166). There are no further relations.

If \mathcal{S} is the group that $[\mathcal{S}_{ij}]$ generate, we have the semi-direct product structure

$$\pi_1(\mathcal{Q}^{(2)}) = \left\{ \mathcal{S} \rtimes \left(\pi_1(\mathcal{Q}^{(1)}) \times \pi_1(\mathcal{Q}^{(1)}) \right) \right\} \rtimes S_2 \quad (168)$$

Here $G_1 \rtimes G_2$ is the semi-direct product of G_1 and G_2 with G_1 being the invariant subgroup. Also $\pi_1(\mathcal{Q}^{(1)})$ acts trivially on \mathcal{S} .

Eq. (168) is to be compared with the corresponding equation for the mapping class group $D^{(2)\infty}/D_0^{(2)\infty}$

of two identical geons [22, 23] if \mathcal{S} is its group of slides,

$$D^{(2)\infty}/D_0^{(2)\infty} = \left\{ \mathcal{S} \rtimes \left(D^{(1)\infty}/D_0^{(1)\infty} \times D^{(1)\infty}/D_0^{(1)\infty} \right) \right\} \rtimes S_2 \quad (169)$$

D. *The case of two identical oriented unknots*

Orienting the knots reduces $\pi_1(\mathcal{Q}^{(1)})$ to $\{e\}$. With that in mind, we can repeat the above discussion (with \bar{q} chosen analogously to above) to find

$$\pi_1(Q^{(2)}) = \mathcal{S} \rtimes S_2 \quad . \quad (170)$$

Some discussion about the quantum theory of these unknots and their unusual statistical features can be found in [15, 32].

XII. FINAL REMARKS

Topological geons were discovered by Friedman and Sorkin. In this paper, we have developed an approach for twisting the spacetimes of topological geons. The twists we consider are localised on the geons and are of the order of Planck scales. They lead to spacetime noncommutativity only on these scales as required by Doplicher, Fredenhagen and Roberts [1]. When geons emerge from asymptotically flat spacetimes, these twists do not seem to affect the Poincaré symmetry for the Wightman functions [26]. But that need not be the case for scattering amplitudes [34].

A significant new contribution of this paper is the extension of Drinfel'd twists to discrete groups and the construction of the associated quantum field theories.

Generic Drinfel'd twists lead to nonassociative spacetimes (such as in [28]) and their quantum fields. For geons, in our approach, such nonassociativity is manifest only at Planck scales. We have reasons to think that Pauli principle violation is a feature of these quantum field theories. While we do not elaborate on this point here, we do plan to study this issue later.

ACKNOWLEDGMENTS

It is a pleasure for Balachandran, Marmo and Martone to thank Alberto Ibort and the Universidad Carlos III de Madrid for their wonderful hospitality and support.

The work of Balachandran and Martone was supported in part by DOE under the grant number DE-FG02-85ER40231 by the Department of Science and Technology (India) and by the Institute of Mathematical Sciences, Chennai. We thank Professor T. R. Govindarajan for his very friendly hospitality at the Institute of Mathematical Sciences, Chennai. Balachandran was also supported by the Department of Science and Technology, India.

A. Ibort would like to acknowledge the partial support from MICINN research project MTM2007-62478 and QUITMAD.

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